

STRONG SOLUTIONS TO THE 3D PRIMITIVE EQUATIONS WITH ONLY HORIZONTAL DISSIPATION: NEAR H^1 INITIAL DATA

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ABSTRACT. In this paper, we consider the initial-boundary value problem of the three-dimensional primitive equations for oceanic and atmospheric dynamics with only horizontal viscosity and horizontal diffusivity. We establish the local, in time, well-posedness of strong solutions, for any initial data $(v_0, T_0) \in H^1$, by using the local, in space, type energy estimate. We also establish the global well-posedness of strong solutions for this system, with any initial data $(v_0, T_0) \in H^1 \cap L^\infty$, such that $\partial_z v_0 \in L^m$, for some $m \in (2, \infty)$, by using the logarithmic type anisotropic Sobolev inequality and a logarithmic type Gronwall inequality. This paper improves the previous results obtained in [Cao, C.; Li, J.; Titi, E. S.: *Global well-posedness of the 3D primitive equations with only horizontal viscosity and diffusivity*, Comm. Pure Appl. Math., **69** (2016), 1492–1531.], where the initial data (v_0, T_0) was assumed to have H^2 regularity.

1. INTRODUCTION

In the context of the large-scale oceanic and atmospheric dynamics, an important feature is that the vertical scale (10–20 kilometers) is much smaller than the horizontal scales (several thousands of kilometers), and therefore, the aspect ratio, i.e. the ratio of the depth (or height) to the horizontal width, is very small. Due to this fact, by the scale analysis (see, e.g., Pedlosky [41]), or taking the small aspect ratio limit to the Navier-Stokes equations (see Azérad–Guillén [1] and Li–Titi [33, 35] for the mathematical justification of this limit), one obtains the primitive equations. The primitive equations form a fundamental block in models for planetary oceanic and atmospheric dynamics, and are widely used in the models of the weather prediction, see, e.g., the books by Haltiner–Williams [21], Lewandowski [29], Majda [40], Pedlosky [41], Vallis [46], Washington–Parkinson [47] and Zeng [49]. Moreover, in the oceanic and atmospheric dynamics, due to the strong horizontal turbulent mixing, the horizontal viscosity and diffusivity are much stronger than the vertical viscosity and diffusivity, respectively.

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In this paper, we consider the following version of the primitive equations for oceanic and atmospheric dynamics, which have only horizontal dissipation, i.e. with only horizontal viscosity and horizontal diffusivity

$$\partial_t v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H p - \Delta_H v + f_0 k \times v = 0, \quad (1.1)$$

$$\partial_z p + T = 0, \quad (1.2)$$

$$\nabla_H \cdot v + \partial_z w = 0, \quad (1.3)$$

$$\partial_t T + (v \cdot \nabla_H) T + w \partial_z T - \Delta_H T = 0, \quad (1.4)$$

where the horizontal velocity $v = (v^1, v^2)$, the vertical velocity w , the temperature T and the pressure p are the unknowns, and f_0 is the Coriolis parameter. The notations $\nabla_H = (\partial_x, \partial_y)$ and $\Delta_H = \partial_x^2 + \partial_y^2$ are the horizontal gradient and the horizontal Laplacian, respectively. Notably, the above system has been first studied by the authors in [10], where the global existence of strong solutions were established, for arbitrary initial data with H^2 regularity; the aim of the present paper is to relax the conditions on the initial data, without losing the global well-posedness of strong solutions.

The mathematical studies of the primitive equations were started by Lions–Temam–Wang [37–39] in the 1990s, where among other issues, global existence of weak solutions was established; however, the uniqueness of weak solutions is still an open question, even for the two-dimensional case. Note that this is different from the incompressible Navier-Stokes equations, as it is well-known that the weak solutions to the two-dimensional incompressible Navier-Stokes equations are unique, see, e.g., Constantin–Foias [15], Ladyzhenskaya [28], Temam [45] and more recently Bardos et al. [2] for the uniqueness of weak solutions, within the class of three-dimensional Leray–Hopf weak solutions, with initial data that are functions of only two spatial variables. However, we would like to point out that, though the general uniqueness of weak solutions to the primitive equations is still unknown, some particular cases have been solved, see [3, 25, 34, 42, 44], and in particular, it is proved in [34] that weak solutions, with bounded initial data, to the primitive equations are unique, as long as the discontinuity of the initial data is sufficiently small. Remarkably, different from the three-dimensional Navier-Stokes equations, global existence and uniqueness of strong solutions to the three-dimensional primitive equations has already been known since the breakthrough work by Cao–Titi [12]. This global existence of strong solutions to the primitive equations were also proved later by Kobelkov [24] and Kukavica–Ziane [26, 27], by using some different approaches, see also Hieber–Kashiwabara [22] and Hieber–Hussien–Kashiwabara [23] for some generalizations in the L^p settings, and Coti Zelati et al. [16] and Guo–Huang [19, 20] for the primitive equations coupled with the moisture equations.

Recall that in the oceanic and atmospheric dynamics, due to the strong horizontal turbulent mixing, the horizontal viscosity and diffusivity are much stronger than the vertical viscosity and diffusivity, respectively, and the vertical viscosity and diffusivity

are very weak. While in all the papers mentioned above, the systems considered are assumed to have both full viscosity and full diffusivity. These lead to the studies of the primitive equations with partial viscosity or partial diffusivity, which have been carried out by Cao–Titi in [13], and by Cao–Li–Titi in [8–11], and see also the survey paper by Li–Titi [33]. In particular, the results in [10, 11] show that the vertical viscosity is not necessary for the global existence of strong solutions to the primitive equations, which is consistent with the physical case (the vertical viscosity for the large scale atmosphere is weak). However, on the other hand, the inviscid primitive equation, with or without coupling to the heat equation has been shown by Cao et al. [7] to blow up in finite time (see also Wong [48]). Combining the global existence results in [10, 11] and the finite-time blow up results in [7, 48], one can conclude that the horizontal viscosity plays an essential role in stabilizing the flow in the large-scale atmosphere and ocean. This provides the mathematical evidences that, in the study of the large scale atmospheric and oceanic dynamics, one can not ignore the eddy viscosity in the horizontal direction, created by the strong horizontal turbulent mixing.

We also note that, for the primitive equations with full viscosity and full diffusivity, the global existence of strong solutions are established for any initial data in H^1 (see [12, 24, 26, 27]), while for the primitive equations with partial viscosity or partial diffusivity, caused by the loss partial viscosities or partial diffusivity, the global strong solutions are established for initial data in H^2 (see [8–10]) or some space weaker than H^2 but stronger than H^1 (see [11]). Compared the results for the primitive equations with partial viscosity or partial diffusivity [8–11] and those with both full viscosity and full diffusivity [12, 24, 26, 27], one may expect to also establish the H^1 theory for the primitive equations with partial viscosity or partial diffusivity, and this paper is devoted to some studies in this direction.

In this paper, we continue the study of the primitive equations with both horizontal viscosity and horizontal diffusivity, which has been studied in [10]. The aim of this paper is to improve the results in [10], and in particular, we want to find the initial data space as weak as possible to guarantee the global existence of strong solutions to system (1.1)–(1.4). Recalling that it is the space H^1 that the initial data is taken from to establish the global existence of strong solutions to the primitive equations with full dissipation, a natural candidate of the initial data spaces is H^1 for the global existence of strong solutions to system (1.1)–(1.4). As it will be shown in this paper, it is the case for the local well-posedness, in other words, for any initial data in H^1 , there is a unique local strong solution to system (1.1)–(1.4), subject to some appropriate boundary and initial conditions; however, due to the lack of the vertical viscosity and the strongly nonlinear term $w\partial_z v$ (which is eventually quadratic in ∇v), the merely H^1 regularity is not sufficient for us to obtain the global strong solutions. Nevertheless, we can prove in this paper that a slightly better space than H^1 is sufficient to guarantee the global existence of strong solutions. More precisely, we prove that for any initial data $(v_0, T_0) \in H^1 \cap L^\infty$, with $\partial_z v_0 \in L^m$, for some

$m \in (2, \infty)$, there is a global strong solution to system (1.1)–(1.4), subject to some appropriate boundary conditions.

We consider the problem in the domain $\Omega_0 = M \times (-h, 0)$, with $M = (0, 1) \times (0, 1)$, and supplement system (1.1)–(1.4) with the following boundary and initial conditions:

$$v, w \text{ and } T \text{ are periodic in } x \text{ and } y, \text{ and of periods } 1, \quad (1.5)$$

$$(\partial_z v, w)|_{z=-h,0} = (0, 0), \quad T|_{z=-h} = 1, \quad T|_{z=0} = 0, \quad (1.6)$$

$$(v, T)|_{t=0} = (v_0, T_0). \quad (1.7)$$

System (1.1)–(1.4) defined on $\Omega_0 = M \times (-h, 0)$, subject to the boundary and initial conditions (1.5)–(1.7), is equivalent to the following system defined on the extended domain $\Omega := M \times (-h, h)$ (see, e.g., [8, 9] for the details)

$$\begin{aligned} \partial_t v + (v \cdot \nabla_H) v - \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \partial_z v - \Delta_H v \\ + f_0 k \times v + \nabla_H \left(p_s(x, y, t) - \int_{-h}^z T(x, y, \xi, t) d\xi \right) = 0, \end{aligned} \quad (1.8)$$

$$\int_{-h}^h \nabla_H \cdot v(x, y, \xi, t) d\xi = 0, \quad (1.9)$$

$$\partial_t T + v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \left(\partial_z T + \frac{1}{h} \right) - \Delta_H T = 0, \quad (1.10)$$

subject to the following boundary and initial conditions

$$v \text{ and } T \text{ are periodic in } x, y, z, \quad (1.11)$$

$$v \text{ and } T \text{ are even and odd in } z, \text{ respectively}, \quad (1.12)$$

$$(v, T)|_{t=0} = (v_0, T_0). \quad (1.13)$$

It should be noticed that in (1.11), as well as in all the cases of periodic boundary conditions below, the periods in x, y are 1, while that in z is $2h$.

Throughout this paper, we use $L^q(\Omega)$, $L^q(M)$ and $W^{m,q}(\Omega)$, $W^{m,q}(M)$ to denote the standard Lebesgue and Sobolev spaces, respectively. For $q = 2$, we use H^m instead of $W^{m,2}$. For simplicity, we still use the notations L^p and H^m to denote the N product spaces $(L^p)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_p$ to denote the L^p norm of u . We denote by \mathbf{x}^H a point in \mathbb{R}^2 . For $0 < r < \infty$, we use $D_r(\mathbf{x}^H)$ to denote an open disk in \mathbb{R}^2 of radius r centered at \mathbf{x}^H . We always use D_r to stand for the disk centered at the origin, except when otherwise explicitly mentioned.

Definition 1.1. *Given a positive number \mathcal{T} . Let $v_0, T_0 \in H^1(\Omega)$ be two spatially periodic functions, such that they are even and odd in z , respectively. A pair (v, T) is called a strong solution to system (1.8)–(1.13) on $\Omega \times (0, \mathcal{T})$ if*

- (i) *v and T are spatially periodic, and they are even and odd in z , respectively;*
- (ii) *v and T have the following regularity properties*

$$\begin{aligned} v, T &\in L^\infty(0, \mathcal{T}; H^1(\Omega)) \cap C([0, \mathcal{T}]; L^2(\Omega)), \\ \nabla_H v, \nabla_H T &\in L^2(0, \mathcal{T}; H^1(\Omega)), \quad \partial_t v, \partial_t T \in L^2(\Omega \times (0, \mathcal{T})); \end{aligned}$$

(iii) v and T satisfy equations (1.8)–(1.10) a.e. in $\Omega \times (0, \mathcal{T})$ and the initial condition (1.13).

Definition 1.2. *The pair (v, T) is called a global strong solution to system (1.8)–(1.13) if it is a strong solution on $\Omega \times (0, \mathcal{T})$, for any $\mathcal{T} \in (0, \infty)$.*

Our main result is concerning the local and global well-posedness of strong solutions to system (1.8)–(1.10), subject to (1.11)–(1.13), as stated in the following:

Theorem 1.1. *Suppose that the periodic functions $v_0, T_0 \in H^1(\Omega)$ are even and odd in z , respectively, with $\int_{-h}^h \nabla_H \cdot v_0(x, y, z) dz = 0$, for any $(x, y) \in M$. Then, there is a unique local, in time, strong solution (v, T) to system (1.8)–(1.10), subject to the boundary and initial conditions (1.11)–(1.13).*

Moreover, if we assume in addition that

$$\partial_z v_0 \in L^m(\Omega), \quad (v_0, T_0) \in L^\infty(\Omega),$$

for some $m \in (2, \infty)$, then the corresponding local strong solution (v, T) can be extended uniquely to be a global one.

Since we consider the system with only horizontal dissipation and the initial data is taken to belong only to H^1 , the arguments used in [18] (with H^1 initial data but for full dissipation case) and [8–11] (for partial dissipation case but with H^2 initial data) do not apply here in order to show the short time existence. Actually, as it will be explained below, applying the standard energy approach to system (1.8)–(1.10) does not yield the required H^1 estimates, even locally in time. The crucial step to prove the local existence of strong solutions is to obtain a local in time estimate for $\partial_z v$. One may try to use the standard energy approach to get such an estimate; however, by doing that, one will encounter a term on the right-hand side of the energy inequality which can not be controlled by the quantities on the left-hand side, unless we have some additional smallness conditions. To overcome this difficulty, we employ a local in space type energy inequality instead of the global in space type. The key idea of the local in space type energy inequality is that initially the local in space integral norms can be as small as desired, which is guaranteed by the absolute continuity of integrals, and we can expect that they will remain small for a short time. As we will see in Proposition 3.2, we can successfully achieve the expected estimate for $\partial_z v$ by using the local in space type energy inequality. Moreover, based on this estimate, we can obtain other relevant estimates which are sufficient to prove the local, in time, existence of strong solutions.

To prove the global existence of strong solutions, we adopt the ideas employed in [10, 11]. The key issue is to derive an $L^\infty(0, \mathcal{T}; L^2(\Omega))$ estimate for $\partial_z v$ for any positive finite time \mathcal{T} . To this end, due to the absence of the vertical viscosity in the momentum equations, we have to derive some control on $\|v\|_\infty^2$, which appears as a

factor in the energy inequalities of the form (see Proposition 4.2)

$$\frac{d}{dt}A(t) + B(t) \leq C\|v\|_\infty^2 A(t) + \text{other terms},$$

where A involves $\|\partial_z v\|_2$. The treatment on $\|v\|_\infty^2$ is similar to that in [10, 11]. More precisely, thanks to the estimates on the growth of the L^q norms of v (Proposition 4.1) and the logarithmic type Sobolev embedding inequality (Lemma 2.4), such $\|v\|_\infty^2$ can be controlled by $\log(A(t) + B(t))$, and as a result, by the logarithmic type Gronwall inequality (Lemma 2.5), we can obtain the desired estimate.

The rest of this paper is arranged as follows: in the next section, section 2, we collect some preliminary results which will be used in the subsequent sections; in section 3, we prove the local well-posedness part of Theorem 1.1, where the local in space energy inequality will be employed; in section 4, we prove the global existence part of Theorem 1.1, where the logarithmic type anisotropic Sobolev inequality and the logarithmic type Gronwall inequality will be employed; in the last Appendix section, we give the proof of a logarithmic type anisotropic Sobolev inequality, a generalization of the Brezis-Gallouet-Wainger inequality [4, 5].

2. PRELIMINARIES

In this section, we collect some preliminary results which will be used in the rest of this paper.

Lemma 2.1. *Let S be a bounded domain in \mathbb{R}^2 , and ϕ, φ, ψ are functions defined on $S \times (-h, h)$. Denote by L the diameter of S . Then, it holds that*

$$\begin{aligned} & \int_S \left(\int_{-h}^h |\phi(x, y, z)| dz \right) \left(\int_{-h}^h |\varphi(x, y, z) \psi(x, y, z)| dz \right) dx dy \\ & \leq Ch^{\frac{1}{2}} \min \left\{ \|\phi\|_2 \|\varphi\|_2^{1/2} \left(\frac{\|\varphi\|_2}{L} + \|\nabla_H \varphi\|_2 \right)^{\frac{1}{2}} \|\psi\|_2^{\frac{1}{2}} \left(\frac{\|\psi\|_2}{L} + \|\nabla_H \psi\|_2 \right)^{\frac{1}{2}}, \right. \\ & \quad \left. \|\phi\|_2^{1/2} \left(\frac{\|\phi\|_2}{L} + \|\nabla_H \phi\|_2 \right)^{\frac{1}{2}} \|\varphi\|_2^{1/2} \left(\frac{\|\varphi\|_2}{L} + \|\nabla_H \varphi\|_2 \right)^{\frac{1}{2}} \|\psi\|_2 \right\}, \end{aligned} \quad (2.1)$$

$$\begin{aligned} & \int_S \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi \psi| dz \right) dx dy \\ & \leq Ch^{\frac{5}{6}} \|\phi\|_6 \|\varphi\|_2^{\frac{2}{3}} \left(\frac{\|\varphi\|_2}{L} + \|\nabla_H \varphi\|_2 \right)^{\frac{1}{3}} \|\psi\|_2, \end{aligned} \quad (2.2)$$

and

$$\int_S \left(\int_{-h}^h |\phi|^2 dz \right) \left(\int_{-h}^h |\varphi|^2 dz \right) dx dy$$

$$\leq C \|\phi\|_2 \left(\frac{\|\phi\|_2}{L} + \|\nabla_H \phi\|_2 \right) \|\varphi\|_2 \left(\frac{\|\varphi\|_2}{L} + \|\nabla_H \varphi\|_2 \right), \quad (2.3)$$

where C is a constant depending only on the shape of S , but not on its size.

Proof. Note that (2.1) has been included in Lemma 2.1 in [10]. We now consider the proof of (2.2). By the Hölder, (dimensionless) Gagliardo-Nirenberg and Minkowski inequalities, we deduce

$$\begin{aligned} & \int_S \left(\int_{-h}^h |\phi(x, y, z)| dz \right) \left(\int_{-h}^h |\varphi(x, y, z) \psi(x, y, z)| dz \right) dx dy \\ & \leq \int_S \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi|^2 dz \right)^{1/2} \left(\int_{-h}^h |\psi|^2 dz \right)^{1/2} dx dy \\ & \leq \left[\int_S \left(\int_{-h}^h |\phi| dz \right)^6 dx dy \right]^{1/6} \left[\int_S \left(\int_{-h}^h |\varphi|^2 dz \right)^{3/2} dx dy \right]^{1/3} \|\psi\|_2 \\ & \leq C h^{\frac{5}{6}} \|\phi\|_6 \left[\int_{-h}^h \left(\int_S |\varphi|^3 dx dy \right)^{2/3} dz \right]^{1/2} \|\psi\|_2 \\ & \leq C h^{\frac{5}{6}} \|\phi\|_6 \left(\int_{-h}^h \|\varphi\|_{2,S}^{4/3} \left(\frac{\|\varphi\|_{2,S}}{L} + \|\nabla_H \varphi\|_{2,S} \right)^{2/3} dz \right)^{1/2} \|\psi\|_2 \\ & \leq C h^{\frac{5}{6}} \|\phi\|_6 \|\varphi\|_2^{2/3} \left(\frac{\|\varphi\|_2}{L} + \|\nabla_H \varphi\|_2 \right)^{1/3} \|\psi\|_2, \end{aligned}$$

proving (2.2). Similarly

$$\begin{aligned} & \int_S \left(\int_{-h}^h |\phi(x, y, z)|^2 dz \right) \left(\int_{-h}^h |\varphi(x, y, z)|^2 dz \right) dx dy \\ & \leq C \left[\int_S \left(\int_{-h}^h |\phi|^2 dz \right)^2 dx dy \right]^{1/2} \left[\int_S \left(\int_{-h}^h |\varphi|^2 dz \right)^2 dx dy \right]^{1/2} \\ & \leq C \left[\int_{-h}^h \left(\int_S |\phi|^4 dx dy \right)^{1/2} dz \right] \left[\int_{-h}^h \left(\int_S |\varphi|^4 dx dy \right)^{1/2} dz \right] \\ & \leq C \int_{-h}^h \|\phi\|_{2,S} \left(\frac{\|\phi\|_{2,S}}{L} + \|\nabla_H \phi\|_{2,S} \right) dz \\ & \quad \times \int_{-h}^h \|\varphi\|_{2,S} \left(\frac{\|\varphi\|_{2,S}}{L} + \|\nabla_H \varphi\|_{2,S} \right) dz \\ & \leq C \|\phi\|_2 \left(\frac{\|\phi\|_2}{L} + \|\nabla_H \phi\|_2 \right) \|\varphi\|_2 \left(\frac{\|\varphi\|_2}{L} + \|\nabla_H \varphi\|_2 \right), \end{aligned}$$

proving (2.3). \square

As a directly corollary of Lemma 2.1, noticing that all discs have the same shape, we have the following lemma.

Lemma 2.2. *Let D_r be an arbitrary disk of radius r in \mathbb{R}^2 , then we have*

$$\begin{aligned} & \int_{D_r} \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi \psi| dz \right) dx dy \\ & \leq Ch^{\frac{1}{2}} \min \left\{ \|\phi\|_2 \|\varphi\|_2^{\frac{1}{2}} \left(\frac{\|\varphi\|_2}{r} + \|\nabla_H \varphi\|_2 \right)^{\frac{1}{2}} \|\psi\|_2^{\frac{1}{2}} \left(\frac{\|\psi\|_2}{r} + \|\nabla_H \psi\|_2 \right)^{\frac{1}{2}}, \right. \\ & \quad \left. \|\phi\|_2^{\frac{1}{2}} \left(\frac{\|\phi\|_2}{r} + \|\nabla_H \phi\|_2 \right)^{\frac{1}{2}} \|\varphi\|_2^{\frac{1}{2}} \left(\frac{\|\varphi\|_2}{r} + \|\nabla_H \varphi\|_2 \right)^{\frac{1}{2}} \|\psi\|_2 \right\}, \end{aligned}$$

and

$$\begin{aligned} & \int_{D_r} \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi \psi| dz \right) dx dy \\ & \leq Ch^{\frac{5}{6}} \|\phi\|_6 \|\varphi\|_2^{\frac{2}{3}} \left(\frac{\|\varphi\|_2}{r} + \|\nabla_H \varphi\|_2 \right)^{\frac{1}{3}} \|\psi\|_2, \end{aligned}$$

for an absolute constant C .

The following lemma is a Sobolev embedding inequality in terms of mixed norm L^p spaces, see Li–Xin [36] for a similar result.

Lemma 2.3. *Let S be a bounded subset of \mathbb{R}^2 . Denote by L the diameter of S . Then, for any function f defined on $S \times (-h, h)$, it holds that*

$$\begin{aligned} \sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{2,S} & \leq \|f\|_2^{1/2} \left(\frac{\|f\|_2}{2h} + 2\|\partial_z f\|_2 \right)^{1/2}, \\ \sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{4,S} & \leq C \left(\frac{\|f\|_2}{h} + \|\partial_z f\|_2 \right)^{1/2} \left(\frac{\|f\|_2}{L} + \|\nabla_H f\|_2 \right)^{1/2}, \end{aligned}$$

where C is a positive constant depending only on the shape of S . In particular, if $S = D_r$, an arbitrary disk of radius r in \mathbb{R}^2 , then one has

$$\sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{L^4(D_r)} \leq C \left(\frac{\|f\|_2}{r} + \|\nabla_H f\|_2 \right)^{1/2} \left(\frac{\|f\|_2}{h} + \|\partial_z f\|_2 \right)^{1/2},$$

for an absolute positive constant C .

Proof. Using the fact that $|k(z)| \leq \frac{1}{2h} \int_{-h}^h |k(\xi)| d\xi + \int_{-h}^h |k'(\xi)| d\xi$, for any $z \in (-h, h)$, one has

$$\sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{2,S}^2 \leq \frac{1}{2h} \int_{-h}^h \|f\|_{2,S}^2 dz + \int_{-h}^h \left| \frac{d}{dz} \|f\|_{2,S}^2 \right| dz$$

$$\leq \frac{\|f\|_2^2}{2h} + 2 \int_{\Omega} |f| |\partial_z f| dx dy dz \leq \|f\|_2 \left(\frac{\|f\|_2}{2h} + 2 \|\partial_z f\|_2 \right), \quad (2.4)$$

proving the first conclusion.

Using again the fact that $|k(z)| \leq \frac{1}{2h} \int_{-h}^h |k(\xi)| d\xi + \int_{-h}^h |k'(\xi)| d\xi$, for all $z \in (-h, h)$, it follows from the (dimensionless) Gagliardo-Nirenberg and Cauchy-Schwarz inequalities that

$$\begin{aligned} \sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{4,S}^4 &\leq \frac{1}{2h} \int_{-h}^h \|f\|_{4,S}^4 dz + \int_{-h}^h \left| \frac{d}{dz} \|f\|_{4,S}^4 \right| dz \\ &\leq \frac{\|f\|_4^4}{2h} + 4 \int_{\Omega} |f|^3 |\partial_z f| dx dy dz \leq \frac{\|f\|_4^4}{2h} + 4 \int_{-h}^h \|f\|_{6,S}^3 \|\partial_z f\|_{2,S} dz \\ &\leq \frac{\|f\|_4^4}{2h} + C \int_{-h}^h \|f\|_{4,S}^2 \left(\frac{\|f\|_{2,S}}{L} + \|\nabla_H f\|_{2,S} \right) \|\partial_z f\|_{2,S} dz \\ &\leq \frac{\|f\|_4^4}{2h} + C \left(\sup_{-h \leq z \leq h} \|f\|_{4,S}^2 \right) \int_{-h}^h \left(\frac{\|f\|_{2,S}}{L} + \|\nabla_H f\|_{2,S} \right) \|\partial_z f\|_{2,S} dz \\ &\leq \frac{1}{2} \sup_{-h \leq z \leq h} \|f\|_{4,S}^4 + C \left[\frac{\|f\|_4^4}{h} + \left(\frac{\|f\|_2^2}{L^2} + \|\nabla_H f\|_2^2 \right) \|\partial_z f\|_2^2 \right], \end{aligned}$$

and thus

$$\sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{4,S}^4 \leq C \left[\frac{\|f\|_4^4}{h} + \left(\frac{\|f\|_2^2}{L^2} + \|\nabla_H f\|_2^2 \right) \|\partial_z f\|_2^2 \right]. \quad (2.5)$$

We estimate $\|f\|_4^4$, on the right-hand side of the above inequality, as follows. By the (dimensionless) two-dimensional Ladyzhenskaya inequality, it follows from (2.4) that

$$\begin{aligned} \|f\|_4^4 &= \int_{-h}^h \|f\|_{4,S}^4 dz \leq C \int_{-h}^h \|f\|_{2,S}^2 \left(\frac{\|f\|_{2,S}}{L} + \|\nabla_H f\|_{2,S} \right)^2 dz \\ &\leq C \left(\sup_{-h \leq z \leq h} \|f\|_{2,S}^2 \right) \left(\frac{\|f\|_2^2}{L^2} + \|\nabla_H f\|_2^2 \right) \\ &\leq C \|f\|_2 \left(\frac{\|f\|_2}{h} + \|\partial_z f\|_2 \right) \left(\frac{\|f\|_2^2}{L^2} + \|\nabla_H f\|_2^2 \right). \end{aligned}$$

Substituting the above inequality into (2.5) and using the Cauchy-Schwarz inequality yield

$$\begin{aligned} \sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{4,S}^4 &\leq C \left[\|\partial_z f\|_2^2 + \frac{\|f\|_2}{h} \left(\frac{\|f\|_2}{h} + \|\partial_z f\|_2 \right) \right] \left(\frac{\|f\|_2^2}{L^2} + \|\nabla_H f\|_2^2 \right) \\ &\leq C \left(\frac{\|f\|_2^2}{h^2} + \|\partial_z f\|_2^2 \right) \left(\frac{\|f\|_2^2}{L^2} + \|\nabla_H f\|_2^2 \right). \end{aligned}$$

which implies

$$\sup_{-h \leq z \leq h} \|f(\cdot, z)\|_{4,S} \leq C \left(\frac{\|f\|_2}{h} + \|\partial_z f\|_2 \right)^{1/2} \left(\frac{\|f\|_2}{L} + \|\nabla_H f\|_2 \right)^{1/2},$$

proving the second conclusion. The third conclusion is a straightforward corollary of the second one, as all the discs have the same shape. This completes the proof. \square

We also need the following logarithmic type anisotropic Sobolev embedding inequality, which generalizes that in [10] to the anisotropic case, see Cao–Farhat–Titi [6], Cao–Wu [14] and Danchin–Paicu [17] for some related inequalities in the isotropic setting in the 2D case. Some similar anisotropic inequality was used by Li–Titi [30] in the study of the Boussinesq equations to relax the assumptions on the initial data.

Lemma 2.4. *Let $\mathbf{p} = (p_1, p_2, p_3)$, with $p_i \in (1, \infty)$, and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$. Then, for any periodic function F on Ω , we have*

$$\|F\|_\infty \leq C_{\mathbf{p}, \lambda, \Omega} \max \left\{ 1, \sup_{r \geq 2} \frac{\|F\|_r}{r^\lambda} \right\} \log^\lambda \left(\sum_{i=1}^3 (\|F\|_{p_i} + \|\partial_i F\|_{p_i}) + e \right),$$

for any $\lambda > 0$.

Proof. Extending F periodically to the whole space. Take a function $\phi \in C_0^\infty(\mathbb{R}^3)$, such that $\phi \equiv 1$ on Ω , and $0 \leq \phi \leq 1$ on \mathbb{R}^3 . Set $f = F\phi$. Noticing that

$$\|F\|_\infty \leq \|f\|_{L^\infty(\mathbb{R}^3)}, \quad \|f\|_{L^r(\mathbb{R}^3)} \leq C\|F\|_r, \quad \|\partial_i f\|_{L^r(\mathbb{R}^3)} \leq C(\|F\|_r + \|\partial_i F\|_r),$$

it follows from Lemma 5.1 (see the Appendix) that

$$\begin{aligned} \|F\|_\infty &\leq \|f\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C_{\mathbf{p}, \lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|f\|_{L^r(\mathbb{R}^3)}}{r^\lambda} \right\} \log^\lambda \left(\sum_{i=1}^3 (\|f\|_{L^{p_i}(\mathbb{R}^3)} + \|\partial_i f\|_{L^{p_i}(\mathbb{R}^3)}) + e \right) \\ &\leq C_{\mathbf{p}, \lambda, \Omega} \max \left\{ 1, \sup_{r \geq 2} \frac{\|F\|_r}{r^\lambda} \right\} \log^\lambda \left(\sum_{i=1}^3 (\|F\|_{p_i} + \|\partial_i F\|_{p_i}) + e \right), \end{aligned}$$

proving the conclusion. \square

We will use the following logarithmic type Gronwall inequality, see Li–Titi [30–32] for some similar type Gronwall inequalities with logarithmic terms.

Lemma 2.5. *Given a positive time $\mathcal{T} \in (0, \infty)$. Let $A(t), B(t)$ and $f(t)$ be nonnegative integrable functions on $[0, \mathcal{T})$, with A being absolutely continuous on $[0, \mathcal{T})$, such that the following holds*

$$A'(t) + B(t) \leq KA(t) \log B(t) + f(t), \quad t \in (0, \mathcal{T}),$$

where $K \geq 1$ is a constant. Then, the following estimate holds

$$A(t) + \int_0^t B(s)ds \leq e^{Q(t)}(1 + 2Q(t)),$$

for any $t \in [0, \mathcal{T})$, where

$$Q(t) = e^{Kt} \left(\log(A(0) + 1) + (2K^2 + 1)t + \int_0^t f(s)ds \right).$$

Proof. Setting

$$A_1(t) = A(t) + 1, \quad B_1(t) = B(t) + A(t) + 1,$$

then, by assumption, we have

$$A'_1(t) + B_1(t) \leq K A_1(t) \log B_1(t) + A_1(t) + f(t).$$

Dividing both sides of the above inequality by $A_1(t)$, noticing that $A_1(t) \geq 1$, one has

$$\begin{aligned} A'_1(t) + \frac{B_1(t)}{A_1(t)} &\leq K \log B_1(t) + f(t) + 1 \\ &= K \log \frac{B_1(t)}{A_1(t)} + K \log A_1(t) + f(t) + 1. \end{aligned}$$

One can easily check that $\log z \leq 2\sqrt{z}$, for any $z \geq 1$, and thus, noticing that $\frac{B_1(t)}{A_1(t)} \geq 1$, we have $\log \frac{B_1(t)}{A_1(t)} \leq 2\sqrt{\frac{B_1(t)}{A_1(t)}}$. Thanks to this estimate, it follows from the above inequality and the Cauchy-Schwarz inequality that

$$\begin{aligned} A'_1(t) + \frac{B_1(t)}{A_1(t)} &\leq 2K \sqrt{\frac{B_1(t)}{A_1(t)}} + K \log A_1(t) + f(t) + 1 \\ &\leq \frac{B_1(t)}{2A_1(t)} + K \log A_1(t) + f(t) + 2K^2 + 1, \end{aligned}$$

that is

$$A'_1(t) + \frac{B_1(t)}{2A_1(t)} \leq K \log A_1(t) + f(t) + 2K^2 + 1.$$

Applying the Gronwall inequality to the above inequality yields

$$\begin{aligned} \log A_1(t) + \int_0^t \frac{B_1(s)}{2A_1(s)} ds &\leq e^{Kt} \left(\log A_1(0) + (2K^2 + 1)t + \int_0^t f(s)ds \right) \\ &= e^{Kt} \left(\log(A(0) + 1) + (2K^2 + 1)t + \int_0^t f(s)ds \right) =: Q(t). \end{aligned}$$

It is obvious that $Q(t)$ is an increasing function of t . Therefore, we deduce

$$A_1(t) + \int_0^t B_1(s)ds = e^{\log A_1(t)} + \int_0^t \frac{B_1(s)}{2A_1(s)} 2e^{\log A_1(s)} ds$$

$$\leq e^{Q(t)} + 2 \int_0^t \frac{B_1(s)}{2A_1(s)} e^{Q(s)} ds \leq e^{Q(t)} (1 + 2Q(t)),$$

which, recalling the definitions of A_1 and B_1 , implies the conclusion. \square

The next lemma is a version of the Aubin-Lions lemma.

Lemma 2.6. *[See Simon [43] Corollary 4] Let $\mathcal{T} \in (0, \infty)$. Assume that X, B and Y are three Banach spaces, with $X \hookrightarrow B \hookrightarrow Y$. Then it holds that*

- (i) *If \mathcal{F} is a bounded subset of $L^p(0, \mathcal{T}; X)$, where $1 \leq p < \infty$, and $\frac{\partial \mathcal{F}}{\partial t} = \{\frac{\partial f}{\partial t} | f \in \mathcal{F}\}$ is bounded in $L^1(0, \mathcal{T}; Y)$, then \mathcal{F} is relatively compact in $L^p(0, \mathcal{T}; B)$;*
- (ii) *If \mathcal{F} is bounded in $L^\infty(0, \mathcal{T}; X)$ and $\frac{\partial \mathcal{F}}{\partial t}$ is bounded in $L^r(0, \mathcal{T}; Y)$, where $r > 1$, then \mathcal{F} is relatively compact in $C([0, \mathcal{T}]; B)$.*

The following proposition, which states the global existence of strong solutions to system with full viscosities, can be proven in the same way as in [9].

Proposition 2.1. *Let $v_0, T_0 \in H^2(\Omega)$ be two periodic functions, such that they are even and odd in z , respectively. Then, for any given $\varepsilon > 0$, there is a unique global strong solution (v, T) to the following system*

$$\begin{aligned} \partial_t v + (v \cdot \nabla_H) v - \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \partial_z v - \Delta_H v - \varepsilon \partial_z^2 v + f_0 k \times v \\ + \nabla_H \left(p_s(x, y, t) - \int_{-h}^z T(x, y, \xi, t) d\xi \right) = 0, \end{aligned} \quad (2.6)$$

$$\int_{-h}^h \nabla_H \cdot v(x, y, \xi, t) d\xi = 0, \quad (2.7)$$

$$\partial_t T + v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \left(\partial_z T + \frac{1}{h} \right) - \Delta_H T - \varepsilon \partial_z^2 T = 0, \quad (2.8)$$

subject to the boundary and initial conditions (1.11)–(1.13), such that

$$\begin{aligned} v &\in L_{loc}^\infty([0, \infty); H^2(\Omega)) \cap C([0, \infty); H^1(\Omega)) \cap L_{loc}^2([0, \infty); H^3(\Omega)), \\ T &\in L_{loc}^\infty([0, \infty); H^2(\Omega)) \cap C([0, \infty); H^1(\Omega)), \quad \nabla_H T \in L_{loc}^2([0, \infty); H^2(\Omega)), \\ \partial_t v, \partial_t T &\in L_{loc}^2([0, \infty); H^1(\Omega)). \end{aligned}$$

3. LOCAL WELL-POSEDNESS OF STRONG SOLUTIONS

In this section, we prove the local well-posedness part of Theorem 1.1. By Proposition 2.1, for any given $\varepsilon > 0$, system (2.6)–(2.8), subject to the boundary and initial conditions (1.11)–(1.13), has a unique global strong solution (v, T) . Next, we are going to establish uniform in ε estimates of this solution.

Proposition 3.1. *Let (v, T) be as in Proposition 2.1. Then the following holds*

$$\sup_{0 \leq s \leq t} (\|v\|_6^2 + \|T\|_6^2)(s) + \int_0^t (\|\nabla_H v\|_2^2 + \varepsilon \|\partial_z v\|_2^2 + \|\nabla_H T\|_2^2)(s) ds \leq K_1(t),$$

for any $t \in [0, \infty)$, where K_1 is a continuous nondecreasing function on $[0, \infty)$, which depends on $\|(v_0, T_0)\|_6$ in a continuous manner, and is independent of ε .

Proof. This is a direct corollary of Proposition 3.1 in [10]. \square

Suppose (v, T) be as in Proposition 2.1, and set $u = \partial_z v$. Then u satisfies the equation

$$\begin{aligned} \partial_t u + (v \cdot \nabla_H)u - \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \partial_z u - \Delta_H u - \varepsilon \partial_z^2 u \\ + f_0 k \times u + (u \cdot \nabla_H)v - (\nabla_H \cdot v)u - \nabla_H T = 0, \end{aligned} \quad (3.1)$$

in $\Omega \times (0, \infty)$.

The following proposition, which gives the estimates on the vertical derivative of the velocity, u , plays the key role in proving the local existence of strong solutions to system (1.8)–(1.13). The basic idea of proving this proposition is the local, in space, energy inequality.

Proposition 3.2. *Let (v, T) be the unique global strong solution to system (2.6)–(2.8), subject to the boundary and initial conditions (1.11)–(1.13).*

There is a suitably small positive constant $\delta_0 \leq 1$, depending only on h , such that if

$$\sup_{\mathbf{x}^H \in M} \int_{-h}^h \int_{D_{2r_0}(\mathbf{x}^H)} |u_0(x, y, z)|^2 dx dy dz \leq \delta_0^2,$$

for some positive number $r_0 \leq 1$, then the following holds true

$$\sup_{0 \leq t \leq t_0^*} \|u\|_2^2(t) + \int_0^{t_0^*} (\|\nabla_H u\|_2^2 + \varepsilon \|\partial_z u\|_2^2)(t) dt \leq C,$$

where C is a constant, depending only on δ_0 and r_0 , and where $t_0^ = \min \left\{ 1, \frac{6r_0^2 \delta_0^2}{C_0} \right\}$, with a positive constant C_0 depending only on h, δ_0 and r_0 .*

Proof. Since u, v and T are periodic, they are defined on the whole space, consequently, equation (3.1) is satisfied on the whole space. For any $\mathbf{x}^H \in M$, set $Q_r(\mathbf{x}^H) = D_r(\mathbf{x}^H) \times (-h, h)$. If \mathbf{x}^H is the original point, we simply use the notation Q_r instead of $Q_r(0)$. Let $\delta_0 \leq 1$ be a sufficiently small positive number, to be determined later. Let $r_0 \leq 1$ be a small enough positive number such that

$$\sup_{\mathbf{x}^H \in M} \int_{-h}^h \int_{D_{2r_0}(\mathbf{x}^H)} |u_0(x, y, z)|^2 dx dy dz \leq \delta_0^2.$$

Set

$$t_0 = \sup \left\{ t \in (0, 1] \left| \sup_{0 \leq s \leq t} \sup_{\mathbf{x}^H \in M} \int_{-h}^h \int_{D_{r_0}(\mathbf{x}^H)} |u(x, y, z, s)|^2 dx dy dz \leq 8\delta_0^2 \right. \right\}.$$

Since any disk of radius $2r_0$ can be covered by finite many, say N , which is independent of r_0 , disks of radius r_0 , one has

$$\sup_{0 \leq t \leq t_0} \sup_{\mathbf{x}^H \in M} \int_{-h}^h \int_{D_{2r_0}(\mathbf{x}^H)} |u(x, y, z)|^2 dx dy dz \leq 8N\delta_0^2. \quad (3.2)$$

Consider a cut-off function $\varphi \in C_0^\infty((D_{2r_0}))$, such that $0 \leq \varphi \leq 1$, $|\nabla_H \varphi| \leq \frac{C}{r_0}$ on D_{2r_0} , with an absolute constant C , and $\varphi \equiv 1$ on D_{r_0} . Multiplying equation (3.1) by $u\varphi^2$ and integrating over Q_{2r_0} , then it follows from integrating by parts that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{Q_{2r_0}} |u|^2 \varphi^2 dx dy dz + \int_{Q_{2r_0}} (\nabla_H u : \nabla_H (u\varphi^2) + \varepsilon \partial_z u \partial_z (u\varphi^2)) dx dy dz \\ &= - \int_{Q_{2r_0}} \left[v \cdot \nabla_H \left(\frac{|u|^2}{2} \right) - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \partial_z \left(\frac{|u|^2}{2} \right) \right] \varphi^2 dx dy dz \\ &+ \int_{Q_{2r_0}} [(u \cdot \nabla_H) v - (\nabla_H \cdot v) u] \cdot u \varphi^2 dx dy dz \\ &- \int_{Q_{2r_0}} \nabla_H T \cdot u \varphi^2 dx dy dz =: I_1 + I_2 + I_3. \end{aligned} \quad (3.3)$$

Recalling that φ is independent of z , it follow from the Cauchy-Schwarz inequality that

$$\begin{aligned} J &:= \int_{Q_{2r_0}} (\nabla_H u : \nabla_H (u\varphi^2) + \varepsilon \partial_z u \partial_z (u\varphi^2)) dx dy dz \\ &= \int_{Q_{2r_0}} [(|\nabla_H u|^2 + \varepsilon |\partial_z u|^2) \varphi^2 + \nabla_H u : u \otimes \nabla_H \varphi^2] dx dy dz \\ &\geq \int_{Q_{2r_0}} [(|\nabla_H u|^2 + \varepsilon |\partial_z u|^2) \varphi^2 - 2 |\nabla_H u| |u| |\varphi| |\nabla_H \varphi|] dx dy dz \\ &\geq \frac{3}{4} \int_{Q_{2r_0}} (|\nabla_H u|^2 + \varepsilon |\partial_z u|^2) \varphi^2 dx dy dz - C \int_{Q_{2r_0}} |u|^2 |\nabla_H \varphi|^2 dx dy dz \\ &\geq \frac{3}{4} \int_{Q_{2r_0}} (|\nabla_H u|^2 + \varepsilon |\partial_z u|^2) \varphi^2 dx dy dz - \frac{C}{r_0^2} \|u\|_{2, Q_{2r_0}}^2. \end{aligned}$$

Integration by parts, and recalling that φ is independent of z , one has

$$I_1 = \int_{Q_{2r_0}} \frac{|u|^2}{2} v \cdot \nabla_H \varphi^2 dx dy dz \leq \int_{Q_{2r_0}} |u|^2 |v| |\varphi| |\nabla_H \varphi| dx dy dz,$$

and

$$\begin{aligned} I_2 &= - \int_{Q_{2r_0}} [(\nabla_H \cdot u)(v \cdot u) \varphi^2 + (u \cdot \nabla_H)(u\varphi^2) \cdot v - v \cdot \nabla_H(|u|^2 \varphi^2)] dx dy dz \\ &\leq 4 \int_{Q_{2r_0}} (|u| |v| |\nabla_H u| \varphi^2 + |u|^2 |v| |\varphi| |\nabla_H \varphi|) dx dy dz. \end{aligned}$$

For I_3 , it follows from integration by parts and the Cauchy-Schwarz inequality that

$$I_3 = \int_{Q_{2r_0}} T \nabla_H \cdot (u\varphi^2) dx dy dz$$

$$\begin{aligned}
&\leq \int_{Q_{2r_0}} |T|(|\nabla_H u| \varphi^2 + 2|u| \varphi |\nabla_H \varphi|) dx dy dz \\
&\leq \frac{1}{4} \int_{Q_{2r_0}} |\nabla_H u|^2 \varphi^2 dx dy dz + C \int_{Q_{2r_0}} (|T|^2 \varphi^2 + |u|^2 |\nabla_H \varphi|^2) dx dy dz \\
&\leq \frac{1}{4} \int_{Q_{2r_0}} |\nabla_H u|^2 \varphi^2 dx dy dz + C \left(\|T\|_{2,Q_{2r_0}}^2 + \frac{\|u\|_{2,Q_{2r_0}}^2}{r_0^2} \right).
\end{aligned}$$

Thanks to the estimates on J , I_1 , I_2 and I_3 , it follows from (3.3) that

$$\begin{aligned}
&\frac{d}{dt} \int_{Q_{2r_0}} |u|^2 \varphi^2 dx dy dz + \int_{Q_{2r_0}} (|\nabla_H u|^2 + \varepsilon |\partial_z u|^2) \varphi^2 dx dy dz \\
&\leq C \int_{Q_{2r_0}} (|u|^2 |v| \varphi |\nabla_H \varphi| + |u| |v| |\nabla_H u| \varphi^2) dx dy dz + C \left(\|T\|_{2,Q_{2r_0}}^2 + \frac{\|u\|_{2,Q_{2r_0}}^2}{r_0^2} \right),
\end{aligned} \tag{3.4}$$

for any $t \in (0, \infty)$.

Using the fact that $|v(x, y, z, t)| \leq \frac{1}{2h} \int_{-h}^h |v(x, y, \xi, t)| d\xi + \int_{-h}^h |\partial_z v(x, y, \xi, t)| d\xi$, and recalling (3.2), it follows from Lemma 2.2, Proposition 3.1, and using the Cauchy-Schwarz and Young inequalities that, for any $t \in [0, t_0]$,

$$\begin{aligned}
\int_{Q_{2r_0}} |u|^2 |v| \varphi |\nabla_H \varphi| dx dy dz &\leq C \int_{Q_{2r_0}} \left(\int_{-h}^h \left(\frac{|v|}{h} + |\partial_z v| \right) d\xi \right) |u|^2 \varphi |\nabla_H \varphi| dx dy dz \\
&= C \int_{D_{2r_0}} \left(\int_{-h}^h \left(\frac{|v|}{h} + |u| \right) d\xi \right) \left(\int_{-h}^h |u|^2 d\xi \right) \varphi |\nabla_H \varphi| dx dy \\
&\leq \frac{C}{r_0} \left(\frac{\|v\|_{2,Q_{2r_0}}}{h} + \|u\|_{2,Q_{2r_0}} \right) \|u\|_{2,Q_{2r_0}} \left(\frac{\|u\|_{2,Q_{2r_0}}}{r_0} + \|\nabla_H u\|_{2,Q_{2r_0}} \right) \\
&\leq \frac{C}{r_0} (1 + \delta_0) \delta_0 \left(\frac{\delta_0}{r_0} + \|\nabla_H u\|_{2,Q_{2r_0}} \right) \leq C \delta_0 \left(\|\nabla_H u\|_{2,Q_{2r_0}}^2 + \frac{1}{r_0^2} \right),
\end{aligned}$$

here we have used that $\|v\|_2$ is bounded, and

$$\begin{aligned}
\int_{Q_{2r_0}} |u| |v| |\nabla_H u| \varphi^2 dx dy dz &\leq C \int_{Q_{2r_0}} \left(\int_{-h}^h \left(\frac{|v|}{h} + |\partial_z v| \right) d\xi \right) |u| |\nabla_H u| dx dy dz \\
&= C \int_{D_{2r_0}} \left(\int_{-h}^h \left(\frac{|v|}{h} + |u| \right) d\xi \right) \left(\int_{-h}^h |u| |\nabla_H u| d\xi \right) dx dy \\
&\leq C \frac{\|v\|_{6,Q_{2r_0}}}{h} \|u\|_{2,Q_{2r_0}}^{2/3} \left(\frac{\|u\|_{2,Q_{2r_0}}^{1/3}}{r_0^{1/3}} + \|\nabla_H u\|_{2,Q_{2r_0}}^{1/3} \right) \|\nabla_H u\|_{2,Q_{2r_0}}
\end{aligned}$$

$$\begin{aligned}
& + C\|u\|_{2,Q_{2r_0}} \left(\frac{\|u\|_{2,Q_{2r_0}}}{r_0} + \|\nabla_H u\|_{2,Q_{2r_0}} \right) \|\nabla_H u\|_{2,Q_{2r_0}} \\
& \leq C \left(\frac{\delta_0}{r_0^{1/3}} + \delta_0^{2/3} \|\nabla_H u\|_{2,Q_{2r_0}}^{1/3} + \frac{\delta_0^2}{r_0} + \delta_0 \|\nabla_H u\|_{2,Q_{2r_0}} \right) \|\nabla_H u\|_{2,Q_{2r_0}} \\
& \leq C \left(\delta_0 \|\nabla_H u\|_{2,Q_{2r_0}}^2 + \frac{1}{r_0^2} \right).
\end{aligned}$$

Substituting the above two inequalities into (3.4), and using (3.2), one obtains

$$\frac{d}{dt} \|u\varphi\|_{2,Q_{2r_0}}^2 + \|\nabla_H u\varphi\|_{2,Q_{2r_0}}^2 + \varepsilon \|\partial_z u\varphi\|_{2,Q_{2r_0}}^2 \leq C\delta_0 \|\nabla_H u\|_{2,Q_{2r_0}}^2 + \frac{C}{r_0^2},$$

and thus

$$\begin{aligned}
& \sup_{0 \leq s \leq t} \|u\varphi\|_{2,Q_{2r_0}}^2 + \int_0^t (\|\nabla_H u\varphi\|_{2,Q_{2r_0}}^2 + \varepsilon \|\partial_z u\varphi\|_{2,Q_{2r_0}}^2) ds \\
& \leq C\delta_0 \int_0^t \|\nabla_H u\|_{2,Q_{2r_0}}^2 ds + \frac{C}{r_0} t,
\end{aligned}$$

for any $t \in [0, t_0]$. Recalling that $\varphi \equiv 1$ on D_{r_0} , this inequality implies

$$\sup_{0 \leq s \leq t} \|u\|_{2,Q_{r_0}}^2 + \int_0^t (\|\nabla_H u\|_{2,Q_{r_0}}^2 + \varepsilon \|\partial_z u\|_{2,Q_{r_0}}^2) ds \leq C\delta_0 \int_0^t \|\nabla_H u\|_{2,Q_{2r_0}}^2 ds + \frac{C}{r_0^2} t,$$

for any $t \in [0, t_0]$.

Similarly, the above inequality still holds true with Q_{r_0} and Q_{2r_0} replaced by $Q_{r_0}(\mathbf{x}^H)$ and $Q_{2r_0}(\mathbf{x}^H)$, respectively. As a result, using the fact that

$$\sup_{\mathbf{x}^H \in M} \int_0^t \|\nabla_H u\|_{2,Q_{2r_0}(\mathbf{x}^H)}^2 ds \leq N \sup_{\mathbf{x}^H \in M} \int_0^t \|\nabla_H u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2 ds,$$

where N (an absolute constant) is the the least number of the disks of radius r_0 that covers a disk of radius $2r_0$, we have

$$\begin{aligned}
& \sup_{\mathbf{x}^H \in M} \left[\sup_{0 \leq s \leq t} \|u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2 + \int_0^t (\|\nabla_H u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2 + \varepsilon \|\partial_z u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2) ds \right] \\
& \leq CN\delta_0 \sup_{\mathbf{x}^H \in M} \left(\int_0^t \|\nabla_H u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2 ds \right) + \frac{C}{r_0^2} t,
\end{aligned}$$

and thus, recalling that δ_0 is sufficiently small, there is a positive constant C_0 depending only on h , such that

$$\sup_{\mathbf{x}^H \in M} \left(\sup_{0 \leq s \leq t} \|u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2 + \int_0^t (\|\nabla_H u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2 + \varepsilon \|\partial_z u\|_{2,Q_{r_0}(\mathbf{x}^H)}^2) ds \right) \leq \frac{C_0}{r_0^2} t \leq 6\delta_0^2, \quad (3.5)$$

for any $t \leq \min \left\{ t_0, \frac{6\delta_0^2 r_0^2}{C_0} \right\}$.

Recalling the definition of t_0 , one has $t_0 \leq 1$. If $t_0 = 1$, then (3.5) holds true for any $t \leq t_0^*$, with

$$t_0^* := \min \left\{ 1, \frac{6\delta_0^2 r_0^2}{C_0} \right\}.$$

If $t_0 < 1$, then by the definition of t_0 and noticing that $u \in C([0, \infty); L^2(\Omega))$, (3.5) implies that

$$\min \left\{ t_0, \frac{6\delta_0^2 r_0^2}{C_2} \right\} < t_0,$$

and thus $t_0 > \frac{6\delta_0^2 r_0^2}{C_0} \geq t_0^*$. Therefore, (3.5) still holds true for any $t \leq t_0^*$. In conclusion, we have estimate (3.5) for any $t \leq t_0^*$. By virtue of estimate (3.5), and covering the domain $M \times (-h, h)$ by finite many Q_{r_0} 's, one obtains the estimate stated in the proposition, and thus completes the proof. \square

Now we give the estimates on the horizontal derivatives of the velocity field.

Proposition 3.3. *Let (v, T) be as in Proposition 2.1. Then one has*

$$\begin{aligned} & \frac{d}{dt} \|\nabla_H v\|_2^2 + (\|\Delta_H v\|_2^2 + \varepsilon \|\nabla_H \partial_z v\|_2^2) \\ & \leq C(\|v\|_2^2 + \|u\|_2^2 + 1)^2 (\|\nabla_H v\|_2^2 + \|\nabla_H u\|_2^2 + 1) \|\nabla_H v\|_2^2, \end{aligned}$$

for any $t \in (0, \infty)$.

Proof. Multiplying equation (2.6) by $-\Delta_H v$, and integrating over Ω , it follows from integrating by parts and using Cauchy-Schwarz and Young inequalities that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla_H v|^2 dx dy dz + \int_{\Omega} (|\Delta_H v|^2 + \varepsilon |\nabla_H \partial_z v|^2) dx dy dz \\ & = \int_{\Omega} \left[(v \cdot \nabla_H) v - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \partial_z v - \nabla_H \left(\int_{-h}^z T d\xi \right) \right] \cdot \Delta_H v dx dy dz \\ & \leq C \int_{\Omega} \left[|v| |\nabla_H v| |\Delta_H v| + \left(\int_{-h}^h |\nabla_H v d\xi| \right) |u| |\Delta_H v| \right] dx dy dz \\ & \quad + \frac{1}{4} \|\Delta_H v\|_2^2 + C \|\nabla_H T\|_2^2. \end{aligned} \tag{3.6}$$

Using the fact that $|\varphi(z)| \leq \frac{1}{2h} \int_{-h}^h |\varphi(\xi)| d\xi + \int_{-h}^h |\partial_z \varphi(\xi)| d\xi$, for every $z \in (-h, h)$, then by Lemma 2.1, and by using the Young inequality, we have the following estimate

$$\begin{aligned} & C \int_{\Omega} |v| |\nabla_H v| |\Delta_H v| dx dy dz \\ & \leq C \int_M \left(\int_{-h}^h \left(\frac{|v|}{h} + |u| \right) d\xi \right) \left(\int_{-h}^h |\nabla_H v| |\Delta_H v| d\xi \right) dx dy \\ & \leq C \left(\frac{\|v\|_2}{h} + \|u\|_2 \right)^{1/2} \left(\frac{\|v\|_2}{h} + \|u\|_2 + \|\nabla_H v\|_2 + \|\nabla_H u\|_2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& \times \|\nabla_H v\|_2^{1/2} (\|\nabla_H v\|_2 + \|\nabla_H^2 v\|_2)^{1/2} \|\Delta_H v\|_2 \\
& \leq \frac{1}{8} \|\Delta_H v\|_2^2 + C[\|\nabla_H v\|_2^2 + (\|v\|_2^2 + \|u\|_2^2)(\|v\|_2^2 + \|u\|_2^2 \\
& \quad + \|\nabla_H v\|_2^2 + \|\nabla_H u\|_2^2) \|\nabla_H v\|_2^2] \\
& \leq \frac{1}{8} \|\Delta_H v\|_2^2 + C(\|v\|_2^2 + \|u\|_2^2 + 1)^2 (\|\nabla_H u\|_2^2 + \|\nabla_H v\|_2^2 + 1) \|\nabla_H v\|_2^2.
\end{aligned}$$

Applying Lemma 2.1 once again, and using the Young inequality, we have

$$\begin{aligned}
& C \int_{\Omega} \left(\int_{-h}^h |\nabla_H v| d\xi \right) |u| |\Delta_H v| dx dy dz \\
& = C \int_M \left(\int_{-h}^h |\nabla_H v| d\xi \right) \left(\int_{-h}^h |u| |\Delta_H v| d\xi \right) dx dy \\
& \leq C \|\nabla_H v\|_2^{1/2} (\|\nabla_H v\|_2 + \|\nabla_H^2 v\|_2)^{1/2} \|u\|_2^{1/2} (\|u\|_2 + \|\nabla_H u\|_2)^{1/2} \|\Delta_H v\|_2 \\
& \leq \frac{1}{8} \|\Delta_H v\|_2^2 + C[\|\nabla_H v\|_2^2 + \|\nabla_H v\|_2^2 \|u\|_2^2 (\|u\|_2^2 + \|\nabla_H u\|_2^2)] \\
& \leq \frac{1}{8} \|\Delta_H v\|_2^2 + C(\|u\|_2^2 + 1)^2 (\|\nabla_H u\|_2^2 + 1) \|\nabla_H v\|_2^2.
\end{aligned}$$

Substituting the above two inequalities into (3.6) yields

$$\begin{aligned}
& \frac{d}{dt} \|\nabla_H v\|_2^2 + (\|\Delta_H v\|_2^2 + \varepsilon \|\nabla_H \partial_z v\|_2^2) \\
& \leq C(\|v\|_2^2 + \|u\|_2^2 + 1)^2 (\|\nabla_H v\|_2^2 + \|\nabla_H u\|_2^2 + 1) \|\nabla_H v\|_2^2,
\end{aligned}$$

proving the conclusion. \square

Estimates on the derivatives of the temperature is stated in the following proposition.

Proposition 3.4. *Let (v, T) be as in Proposition 2.1. Then it holds that*

$$\begin{aligned}
& \frac{d}{dt} \|\nabla T\|_2^2 + \|\nabla_H \nabla T\|_2^2 + \varepsilon \|\partial_z \nabla T\|_2^2 \\
& \leq C(\|v\|_2^2 + \|\nabla v\|_2^2 + 1)^2 (\|\nabla_H v\|_2^2 + \|\nabla_H \nabla v\|_2^2 + 1) (\|\nabla T\|_2^2 + 1),
\end{aligned}$$

for any $t \in (0, \infty)$.

Proof. Multiplying equation (2.8) by $-\Delta T$ and integrating over Ω , then it follows from integrating by parts that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla T|^2 dx dy dz + \int_{\Omega} (|\nabla_H \nabla T|^2 + \varepsilon |\partial_z \nabla T|^2) dx dy dz \\
& = \int_{\Omega} \left[v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \left(\partial_z T + \frac{1}{h} \right) \right] \Delta T dx dy dz
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} \left[v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \left(\partial_z T + \frac{1}{h} \right) \right] \Delta_H T dx dy dz \\
&\quad + \int_{\Omega} \left[v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \left(\partial_z T + \frac{1}{h} \right) \right] \partial_z^2 T dx dy dz \\
&= \int_{\Omega} \left[v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \left(\partial_z T + \frac{1}{h} \right) \right] \Delta_H T dx dy dz \\
&\quad - \int_{\Omega} \left[(\partial_z v \cdot \nabla_H T + v \cdot \nabla_H \partial_z T) \partial_z T - \frac{1}{2} (\nabla_H \cdot v) |\partial_z T|^2 + \frac{\nabla_H \cdot v}{h} \partial_z T \right] dx dy dz \\
&= \int_{\Omega} \left[v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \left(\partial_z T + \frac{1}{h} \right) \right] \Delta_H T dx dy dz \\
&\quad - \int_{\Omega} (u \cdot \nabla_H T \partial_z T + 2v \cdot \nabla_H \partial_z T \partial_z T + \frac{1}{h} \nabla_H \cdot v \partial_z T) dx dy dz \\
&\leq \int_{\Omega} \left[v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \left(\partial_z T + \frac{1}{h} \right) \right] \Delta_H T dx dy dz \\
&\quad - \int_{\Omega} (u \cdot \nabla_H T \partial_z T + 2v \cdot \nabla_H \partial_z T \partial_z T) dx dy dz + C \|\nabla_H v\|_2 \|\partial_z T\|_2. \tag{3.7}
\end{aligned}$$

Using the fact that $|v(z)| \leq \frac{1}{2h} \int_{-h}^h |v(\xi)| d\xi + \int_{-h}^h |\partial_z v(\xi)| d\xi$, for all $z \in (-h, h)$, it follows from Lemma 2.1, and using the Young inequality that

$$\begin{aligned}
&\left| \int_{\Omega} (v \cdot \nabla_H T \Delta_H T - 2v \cdot \nabla_H \partial_z T \partial_z T) dx dy dz \right| \\
&\leq C \int_M \left(\int_{-h}^h \left(\frac{|v|}{h} + |u| \right) d\xi \right) \left(\int_{-h}^h |\nabla T| |\nabla_H \nabla T| d\xi \right) dx dy \\
&\leq C \left(\frac{\|v\|_2}{h} + \|u\|_2 \right)^{1/2} \left(\frac{\|v\|_2}{h} + \|u\|_2 + \|\nabla_H v\|_2 + \|\nabla_H u\|_2 \right)^{1/2} \\
&\quad \times \|\nabla T\|_2^{1/2} (\|\nabla T\|_2 + \|\nabla_H \nabla T\|_2)^{1/2} \|\nabla_H \nabla T\|_2 \\
&\leq \frac{1}{6} \|\nabla_H \nabla T\|_2^2 + C [\|\nabla T\|_2^2 + (\|v\|_2^2 + \|u\|_2^2) (\|v\|_2^2 + \|u\|_2^2 \\
&\quad + \|\nabla_H v\|_2^2 + \|\nabla_H u\|_2^2) \|\nabla T\|_2^2] \\
&\leq \frac{1}{6} \|\nabla_H \nabla T\|_2^2 + C (\|v\|_2^2 + \|u\|_2^2 + 1)^2 (\|\nabla_H v\|_2^2 + \|\nabla_H u\|_2^2 + 1) \|\nabla T\|_2^2.
\end{aligned}$$

Recalling that T is odd and periodic in z , one has $T(x, y, -h, t) = -T(x, y, h, t) = -T(x, y, -h, t)$, and thus $T|_{z=-h, h} = 0$, which implies $\nabla_H T|_{z=-h, h} = 0$. Thanks to this fact, it follows $|\nabla_H T(z)| \leq \int_{-h}^h |\nabla_H \partial_z T| d\xi$, for every $z \in (-h, h)$. Using this

inequality, it follows from Lemma 2.1 and the Young inequality that

$$\begin{aligned}
& \left| \int_{\Omega} u \cdot \nabla_H T \partial_z T dx dy dz \right| \\
& \leq C \int_M \left(\int_{-h}^h |\nabla_H \partial_z T| d\xi \right) \left(\int_{-h}^h |u| |\partial_z T| d\xi \right) dx dy \\
& \leq C \|\nabla_H \partial_z T\|_2 \|u\|_2^{1/2} (\|u\|_2 + \|\nabla_H u\|_2)^{1/2} \|\partial_z T\|_2^{1/2} (\|\partial_z T\|_2 + \|\nabla_H \partial_z T\|_2)^{1/2} \\
& \leq \frac{1}{6} \|\nabla_H \partial_z T\|_2^2 + C[\|\partial_z T\|_2^2 + \|u\|_2^2 (\|u\|_2^2 + \|\nabla_H u\|_2^2) \|\partial_z T\|_2^2] \\
& \leq \frac{1}{6} \|\nabla_H \partial_z T\|_2^2 + C(\|u\|_2^2 + 1)^2 (\|\nabla_H u\|_2^2 + 1) \|\partial_z T\|_2^2.
\end{aligned}$$

Applying Lemma 2.1 once again, and using the Young inequality yields

$$\begin{aligned}
& \left| \int_{\Omega} \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) \partial_z T \Delta_H T dx dy dz \right| \\
& \leq C \int_M \left(\int_{-h}^h |\nabla_H v| d\xi \right) \left(\int_{-h}^h |\partial_z T| |\Delta_H T| d\xi \right) dx dy \\
& \leq C \|\nabla_H v\|_2^{1/2} (\|\nabla_H v\|_2 + \|\nabla_H^2 v\|_2)^{1/2} \|\partial_z T\|_2^{1/2} (\|\partial_z T\|_2 + \|\nabla_H \partial_z T\|_2)^{1/2} \|\Delta_H T\|_2 \\
& \leq \frac{1}{6} \|\nabla_H \nabla T\|_2^2 + C[\|\partial_z T\|_2^2 + \|\nabla_H v\|_2^2 (\|\nabla_H v\|_2^2 + \|\Delta_H v\|_2^2) \|\partial_z T\|_2^2] \\
& \leq \frac{1}{6} \|\nabla_H \nabla T\|_2^2 + C(\|\nabla_H v\|_2^2 + 1)^2 (\|\Delta_H v\|_2^2 + 1) \|\partial_z T\|_2^2.
\end{aligned}$$

Substituting these inequalities into (3.7) leads to

$$\begin{aligned}
& \frac{d}{dt} \|\nabla T\|_2^2 + \|\nabla_H \nabla T\|_2^2 + \varepsilon \|\partial_z \nabla T\|_2^2 \\
& \leq C(\|v\|_2^2 + \|\nabla v\|_2^2 + 1)^2 (\|\nabla_H v\|_2^2 + \|\nabla_H \nabla v\|_2^2 + 1) (\|\nabla T\|_2^2 + 1),
\end{aligned}$$

completing the proof. \square

Proposition 3.5. *Let (v, T) be as in Proposition 2.1. Then we have*

$$\begin{aligned}
\|\partial_t v\|_2^2 + \|\partial_t T\|_2^2 & \leq C[\varepsilon^2 (\|\partial_z u\|_2^2 + \|\partial_z^2 T\|_2^2) + (\|v\|_{H^1}^2 + \|T\|_{H^1}^2 + 1)^2 \\
& \quad \times (\|\nabla_H v\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2 + 1)],
\end{aligned}$$

for any $t \in (0, \infty)$.

Proof. Define functions f_1 and f_2 as

$$\begin{aligned}
f_1 & = - (v \cdot \nabla_H) v + \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \partial_z v \\
& \quad - f_0 k \times v + \nabla_H \left(\int_{-h}^z T(x, y, \xi, t) d\xi \right)
\end{aligned}$$

and

$$f_2 = v \cdot \nabla_H T + \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \left(\partial_z T + \frac{1}{h} \right).$$

Applying Lemma 2.1, and using the Sobolev and Young inequalities, we have the following estimates

$$\begin{aligned} \|f_2\|_2^2 &\leq C \int_{\Omega} \left[|v|^2 |\nabla_H T|^2 + \left(\int_{-h}^h |\nabla_H v| d\xi \right)^2 (1 + |\partial_z T|^2) \right] dx dy dz \\ &\leq C(\|v\|_4^2 \|\nabla_H T\|_4^2 + \|\nabla_H v\|_2^2) + C \int_M \left(\int_{-h}^h |\nabla_H v|^2 d\xi \right) \left(\int_{-h}^h |\partial_z T|^2 d\xi \right) dx dy \\ &\leq C(\|v\|_{H^1}^2 \|\nabla_H T\|_{H^1}^2 + \|\nabla_H v\|_2^2) \\ &\quad + C\|\nabla_H v\|_2(\|\nabla_H v\|_2 + \|\nabla_H^2 v\|_2) \|\partial_z T\|_2(\|\partial_z T\|_2 + \|\nabla_H \partial_z T\|_2) \\ &\leq C(\|v\|_{H^1}^2 \|\nabla_H T\|_{H^1}^2 + \|v\|_{H^1}^2) + C[\|v\|_{H^1}^2(\|v\|_{H^1}^2 + \|\nabla_H v\|_{H^1}^2) \\ &\quad + \|T\|_{H^1}^2(\|T\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2)] \\ &\leq C(\|v\|_{H^1}^2 + \|T\|_{H^1}^2 + 1)^2(\|\nabla_H v\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2 + 1), \end{aligned}$$

and similarly

$$\begin{aligned} \|f_1\|_2^2 &\leq C(\|v\|_{H^1}^2 + 1)^2(\|\nabla_H v\|_{H^1}^2 + 1) + C(\|v\|_2^2 + \|\nabla_H T\|_2^2) \\ &\leq C(\|v\|_{H^1}^2 + \|T\|_{H^1}^2 + 1)^2(\|\nabla_H v\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2 + 1). \end{aligned}$$

Note that v and T satisfies

$$\partial_t v - \Delta_H v - \varepsilon \partial_z^2 v + \nabla_H p_s(x, y, t) = f_1, \quad (3.8)$$

$$\int_{-h}^h \nabla_H \cdot v(x, y, \xi, t) d\xi = 0, \quad (3.9)$$

$$\partial_t T - \Delta_H T - \varepsilon \partial_z^2 T = f_2. \quad (3.10)$$

By (3.10), we have

$$\begin{aligned} \|\partial_t T\|_2^2 &\leq \|\Delta_H T\|_2^2 + \varepsilon^2 \|\partial_z^2 T\|_2^2 + \|f_2\|_2^2 \\ &\leq C(\|v\|_{H^1}^2 + \|T\|_{H^1}^2 + 1)^2(\|\nabla_H v\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2 + 1) + \varepsilon^2 \|\partial_z^2 T\|_2^2. \end{aligned}$$

By the aid of (3.8) and (3.9), one can easily see that

$$\Delta_H p_s(x, y, t) = \frac{1}{2h} \int_{-h}^h \nabla_H \cdot f_1(x, y, \xi, t) d\xi,$$

and thus, by the elliptic estimates, one obtains

$$\begin{aligned} \|\nabla_H p_s\|_{L^2(M)}^2 &\leq C \left\| \int_{-h}^h f_1(x, y, \xi, t) d\xi \right\|_{L^2(M)}^2 \leq C \|f_1\|_2^2 \\ &\leq C(\|v\|_{H^1}^2 + \|T\|_{H^1}^2 + 1)^2(\|\nabla_H v\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2 + 1). \end{aligned}$$

On account of this, it follows from (3.8) that

$$\begin{aligned}\|\partial_t v\|_2^2 &\leq C(\|\Delta_H v\|_2^2 + \varepsilon^2 \|\partial_z^2 v\|_2^2 + \|\nabla_H p_s\|_2^2 + \|f_1\|_2^2) \\ &\leq C[\varepsilon^2 \|\partial_z u\|_2^2 + (\|v\|_{H^1}^2 + \|T\|_{H^1}^2 + 1)^2 (\|\nabla_H v\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2 + 1)].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|\partial_t v\|_2^2 + \|\partial_t T\|_2^2 &\leq C[\varepsilon^2 (\|\partial_z u\|_2^2 + \|\partial_z^2 T\|_2^2) + (\|v\|_{H^1}^2 + \|T\|_{H^1}^2 + 1)^2 \\ &\quad \times (\|\nabla_H v\|_{H^1}^2 + \|\nabla_H T\|_{H^1}^2 + 1)],\end{aligned}$$

completing the proof. \square

By the aid of these propositions, we are now ready to give the proof of local well-posedness part of Theorem 1.1.

Proof of local well-posedness. Consider the periodic functions $v_0^\varepsilon, T_0^\varepsilon \in H^2(\Omega)$, such that they are even and odd in z , respectively, and

$$(v_0^\varepsilon, T_0^\varepsilon) \rightarrow (v_0, T_0), \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } H^1(\Omega).$$

It is obvious that $u_0^\varepsilon \rightarrow u_0$, as $\varepsilon \rightarrow 0$, in $L^2(\Omega)$, where $u_0^\varepsilon = \partial_z v_0^\varepsilon$ and $u_0 = \partial_z v_0$.

Let δ_0 be the constant sated in Proposition 3.2. By the absolutely continuity of the integral, there is a positive number $r_0 \leq 1$, such that

$$\sup_{\mathbf{x}^H \in M} \int_{-h}^h \int_{D_{2r_0}(\mathbf{x}^H)} |u_0(x, y, z)|^2 dx dy dz \leq \frac{\delta_0^2}{2}.$$

Thus, there exists $\varepsilon_0 > 0$, depending on δ_0 , such that for every $\varepsilon \in (0, \varepsilon_0)$, one has

$$\sup_{\mathbf{x}^H \in M} \int_{-h}^h \int_{D_{2r_0}(\mathbf{x}^H)} |u_0^\varepsilon(x, y, z)|^2 dx dy dz \leq \delta_0^2.$$

For any $\varepsilon \in (0, \varepsilon_0)$, let $(v_\varepsilon, T_\varepsilon)$ be the unique strong solution corresponding to the initial data $(v_0^\varepsilon, T_0^\varepsilon)$ as stated by Proposition 2.1. Set $u_\varepsilon = \partial_z v_\varepsilon$. By Proposition 3.1 and Proposition 3.2, we have the estimate

$$\begin{aligned}\sup_{0 \leq t \leq t_0^*} (\|v_\varepsilon\|_2^2 + \|T_\varepsilon\|_2^2 + \|u_\varepsilon\|_2^2) + \int_0^{t_0^*} (\|\nabla_H v_\varepsilon\|_2^2 \\ + \varepsilon \|\partial_z v_\varepsilon\|_2^2 + \|\nabla_H T_\varepsilon\|_2^2 + \|\nabla_H u_\varepsilon\|_2^2 + \varepsilon \|\partial_z u_\varepsilon\|_2^2) ds \leq C,\end{aligned}\tag{3.11}$$

where C is a positive constant and t_0^* is the same constant as in Proposition 3.2, both depend on δ_0 , but are independent of $\varepsilon \in (0, \varepsilon_0)$. On account of the above estimate, by using the Gronwall inequality, one can easily obtain from Proposition 3.3, Proposition 3.4 and Proposition 3.5 that

$$\begin{aligned}\sup_{0 \leq t \leq t_0^*} (\|\nabla_H v_\varepsilon\|_2^2 + \|\nabla_H T_\varepsilon\|_2^2) + \int_0^{t_0^*} (\|\Delta_H v_\varepsilon\|_2^2 \\ + \varepsilon \|\nabla_H \partial_z v_\varepsilon\|_2^2 + \|\nabla_H \nabla_H T_\varepsilon\|_2^2 + \|\partial_t v_\varepsilon\|_2^2 + \|\partial_t T_\varepsilon\|_2^2) \leq C,\end{aligned}$$

which, combined with (3.11), gives

$$\begin{aligned} \sup_{0 \leq t \leq t_0^*} (\|v_\varepsilon\|_{H^1}^2 + \|T_\varepsilon\|_{H^1}^2) + \int_0^{t_0^*} (\|\nabla_H v_\varepsilon\|_{H^1}^2 \\ + \varepsilon \|\partial_z v_\varepsilon\|_{H^1}^2 + \|\nabla_H T_\varepsilon\|_{H^1}^2 + \|\partial_t v_\varepsilon\|_2^2 + \|\partial_t T_\varepsilon\|_2^2) \leq C, \end{aligned} \quad (3.12)$$

for a positive constant C , which depends on δ_0 , but is independent of $\varepsilon \in (0, \varepsilon_0)$. Thanks to the above estimate, by the Aubin-Lions lemma, i.e. Lemma 2.6, there is a subsequence, still denoted by $(v_\varepsilon, T_\varepsilon)$, and (v, T) , such that as $\varepsilon \rightarrow 0$ one has

$$\begin{aligned} (v_\varepsilon, T_\varepsilon) &\rightarrow (v, T), \quad \text{in } C([0, t_0^*]; L^2(\Omega)), \\ \varepsilon \partial_z v_\varepsilon &\rightarrow 0, \quad \text{in } L^\infty(0, t_0^*; L^2(\Omega)), \\ (v_\varepsilon, T_\varepsilon) &\overset{*}{\rightharpoonup} (v, T), \quad \text{in } L^\infty(0, t_0^*; H^1(\Omega)), \\ (\nabla_H v_\varepsilon, \nabla_H T_\varepsilon) &\rightharpoonup (\nabla_H v, \nabla_H T), \quad \text{in } L^2(0, t_0^*; H^1(\Omega)), \\ (\partial_t v_\varepsilon, \partial_t T_\varepsilon) &\rightharpoonup (\partial_t v, \partial_t T), \quad \text{in } L^2(0, t_0^*; L^2(\Omega)), \end{aligned}$$

where \rightharpoonup and $\overset{*}{\rightharpoonup}$ are the weak convergence and weak-* convergence, respectively. Thanks to these convergences, noticing that

$$\begin{aligned} \left(\int_{-h}^z \nabla_H v_\varepsilon d\xi \right) \partial_z v_\varepsilon &= \partial_z \left(\left(\int_{-h}^z \nabla_H \cdot v_\varepsilon d\xi \right) v_\varepsilon \right) - (\nabla_H \cdot v_\varepsilon) v_\varepsilon, \\ \left(\int_{-h}^z \nabla_H T_\varepsilon d\xi \right) \partial_z T_\varepsilon &= \partial_z \left(\left(\int_{-h}^z \nabla_H \cdot T_\varepsilon d\xi \right) T_\varepsilon \right) - (\nabla_H \cdot T_\varepsilon) T_\varepsilon, \end{aligned}$$

one can take the limit $\varepsilon \rightarrow 0$, at the level of the subsequence, to system (2.6)–(2.8) to conclude that (v, T) is a strong solution to system (1.8)–(1.13) on $\Omega \times (0, t_0^*)$.

Recalling the regularity properties of the strong solutions, the uniqueness of strong solutions to system (1.8)–(1.13) is a direct corollary of Proposition 2.4 of [9]. This completes the proof of the local well-posedness part of Theorem 1.1. \square

4. GLOBAL EXISTENCE OF STRONG SOLUTIONS

In this section, we prove that if the initial data $(v_0, T_0) \in H^1$ has the additional regularity that $\partial_z v_0 \in L^m(\Omega)$, for some $m \in (2, \infty)$, and $(v_0, T_0) \in L^\infty(\Omega)$, then the local strong solution established in the previous section can be extended to be a global one, in other words, we will prove the global existence part of Theorem 1.1.

Checking the proof in the previous section, to prove the global existence of strong solutions to system (1.8)–(1.13), it suffices to establish estimate (3.12), for any finite time interval $[0, \mathcal{T}]$, for the global strong solution (v, T) to system (2.6)–(2.8), subject to (1.11)–(1.13). Moreover, by Propositions 3.1, 3.4 and 3.5, to get such an estimate, we only need to prove the $L^\infty(0, \mathcal{T}; L^2(\Omega))$ estimate for $\partial_z v$, for any finite time interval $[0, \mathcal{T}]$.

The following proposition is a straightforward corollary of Proposition 3.1 in [10].

Proposition 4.1. *Let (v, T) be as in Proposition 2.1. Then, for any finite time \mathcal{T} , we have the estimate*

$$\sup_{0 \leq t \leq \mathcal{T}} \sup_{2 \leq q < \infty} \frac{\|v\|_q}{\sqrt{q}} \leq C,$$

where C is a positive constant depending only on h, \mathcal{T} and $\|(v_0, T_0)\|_\infty$, but is independent of ε .

Proposition 4.2. *Let (v, T) be as in Proposition 2.1, and set $u := \partial_z v$. Then, for any finite time \mathcal{T} , we have the following estimates*

$$\frac{d}{dt} \|u\|_q^q + \int_\Omega |u|^{q-2} (|\nabla_H u|^2 + \varepsilon |\partial_z u|^2) dx dy dz \leq C(1 + \|v\|_\infty^2)(1 + \|u\|_q^q),$$

and

$$\begin{aligned} & \frac{d}{dt} \|\nabla_H v\|_2^2 + \|\Delta_H v\|_2^2 + \varepsilon \|\nabla_H \partial_z v\|_2^2 \\ & \leq C \|v\|_\infty^2 \|\nabla_H v\|_2^2 + C(\|u\|_r^{\frac{4r}{r-2}} + \|\nabla_H T\|_2^2 + 1), \end{aligned}$$

for any $q \in [2, \infty)$, $r \in (2, \infty)$ and any $t \in (0, \mathcal{T})$, where C is a positive constant depending only on $q, r, h, \mathcal{T}, \|v_0\|_2$ and $\|T_0\|_\infty$, but is independent of ε .

Proof. The first conclusion has been proved in Proposition 4.1 (i) of [10]. We now prove the second one. Multiplying equation (2.6) by $-\Delta_H v$, and integrating over Ω , then it follows from integrating by parts that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla_H v|^2 dx dy dz + \int_\Omega (|\Delta_H v|^2 + \varepsilon |\nabla_H \partial_z v|^2) dx dy dz \\ & = \int_\Omega \left[(v \cdot \nabla_H) v - \left(\int_{-h}^z \nabla_H \cdot v d\xi \right) u - \nabla_H \left(\int_{-h}^z T d\xi \right) \right] \cdot \Delta_H v dx dy dz \\ & \leq C(\|v\|_\infty \|\nabla_H v\|_2 + \|\nabla_H T\|_2) \|\Delta_H v\|_2 + \int_M \int_{-h}^h |\nabla_H v| dz \int_{-h}^h |u| |\Delta_H v| dz dx dy. \end{aligned}$$

Recalling that $\sup_{0 \leq t \leq \mathcal{T}} \|v\|_2^2 \leq C$, guaranteed by Proposition 3.1, it follows from the Hölder, Minkowski and Gagliardo-Nirenberg inequalities that

$$\begin{aligned} & \int_M \int_{-h}^h |\nabla_H v| dz \int_{-h}^h |u| |\Delta_H v| dz dx dy \\ & \leq \int_M \int_{-h}^h |\nabla_H v| dz \left(\int_{-h}^h |u|^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\Delta_H v|^2 dz \right)^{\frac{1}{2}} dx dy \\ & \leq \left[\int_M \left(\int_{-h}^h |\nabla_H v| dz \right)^{\frac{2r}{r-2}} dx dy \right]^{\frac{r-2}{2r}} \left[\int_M \left(\int_{-h}^h |u|^2 dz \right)^{\frac{r}{2}} dx dy \right]^{\frac{1}{r}} \|\Delta_H v\|_2 \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-h}^h \|\nabla_H v\|_{\frac{2r}{r-2}, M} dz \left(\int_{-h}^h \|u\|_{r, M}^2 dz \right)^{\frac{1}{2}} \|\Delta_H v\|_2 \\
&\leq C \int_{-h}^h \|v\|_{2, M}^{\frac{1}{2} - \frac{1}{r}} (\|v\|_{2, M} + \|\Delta_H v\|_{2, M})^{\frac{1}{2} + \frac{1}{r}} dz \|u\|_r \|\Delta_H v\|_2 \\
&\leq C(\|v\|_2 + \|v\|_2^{\frac{1}{2} - \frac{1}{r}} \|\Delta_H v\|_2^{\frac{1}{2} + \frac{1}{r}}) \|u\|_r \|\Delta_H v\|_2 \\
&\leq C(1 + \|\Delta_H v\|_2^{\frac{1}{2} + \frac{1}{r}}) \|u\|_r \|\Delta_H v\|_2.
\end{aligned}$$

Substitute the above inequality into the previous one and using the Young inequality, we then get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\nabla_H v\|_2^2 + \|\Delta_H v\|_2^2 + \varepsilon \|\nabla_H \partial_z v\|_2^2 \\
&\leq C(\|v\|_\infty \|\nabla_H v\|_2 + \|\nabla_H T\|_2) \|\Delta_H v\|_2 + C(1 + \|\Delta_H v\|_2^{\frac{1}{2} + \frac{1}{r}}) \|u\|_r \|\Delta_H v\|_2 \\
&\leq \frac{1}{2} \|\Delta_H v\|_2^2 + C(\|v\|_\infty^2 \|\nabla_H v\|_2^2 + \|u\|_r^{\frac{4r}{r-2}} + \|\nabla_H T\|_2^2 + 1),
\end{aligned}$$

which implies the conclusion. \square

Thanks to the above two propositions, we can apply the logarithmic type Sobolev embedding inequality (Lemma 2.4) and the logarithmic type Gronwall inequality (Lemma 2.5) to derive the $L^\infty(0, \mathcal{T}; L^2(\Omega))$ estimate on ∇v .

Proposition 4.3. *Let (v, T) be as in Proposition 2.1, and let $m \in (2, \infty)$. Then, for any finite time \mathcal{T} , we have*

$$\sup_{0 \leq t \leq \mathcal{T}} (\|\nabla v\|_2^2 + \|\partial_z v\|_m^m) + \int_0^{\mathcal{T}} (\|\nabla_H \nabla v\|_2^2 + \varepsilon \|\partial_z \nabla v\|_2^2) dt \leq C,$$

for a positive constant C depending only on m, h, \mathcal{T} and $\|v_0\|_{H^1} + \|\partial_z v_0\|_m + \|(v_0, T_0)\|_\infty$, but is independent of ε .

Proof. Given $\mathcal{T} \in (0, \infty)$. Set $u = \partial_z v$, and define

$$\begin{aligned}
A_1(t) &= \|u(t)\|_2^2 + \|u(t)\|_m^m + e, & B_1(t) &= \|\nabla_H u(t)\|_2^2 + \varepsilon \|\partial_z u\|_2^2 + e, \\
A_2(t) &= \|\nabla_H v(t)\|_2^2 + e, & B_2(t) &= \|\Delta_H v(t)\|_2^2 + \varepsilon \|\partial_z \nabla_H v\|_2^2 + e.
\end{aligned}$$

By Proposition 4.2, we have

$$\begin{aligned}
\frac{d}{dt} A_1(t) + B_1(t) &\leq C(1 + \|v(t)\|_\infty^2) A_1(t), \\
\frac{d}{dt} A_2(t) + B_2(t) &\leq C\|v(t)\|_\infty^2 A_2(t) + C A_1^\lambda(t) + C \|\nabla_H T(t)\|_2^2,
\end{aligned}$$

for any $t \in (0, \mathcal{T})$, where $\lambda = \frac{4}{m-2}$, and C is a positive constant depending only on $m, h, \mathcal{T}, \|v_0\|_2$ and $\|T_0\|_\infty$, but independent of ε .

Multiplying the first inequality by $1 + \lambda A_1^{\lambda-1}(t)$, and summing the resulting inequality with the second one yields

$$\begin{aligned} \frac{d}{dt}(A_1(t) + A_1^\lambda(t) + A_2(t)) + (1 + \lambda A_1^{\lambda-1}(t))B_1(t) + B_2(t) \\ \leq C(1 + \|v\|_\infty^2)(A_1(t) + A_1^\lambda(t) + A_2(t)) + C\|\nabla_H T(t)\|_2^2, \end{aligned}$$

for any $t \in (0, \mathcal{T})$, where C is a positive constant depending only on $m, h, \mathcal{T}, \|v_0\|_2$ and $\|T_0\|_\infty$, and is independent of ε .

Summing both sides of the above inequality with $A_1(t) + A_1^\lambda(t) + A_2(t)$, and setting

$$\begin{aligned} A(t) &= A_1(t) + A_1^\lambda(t) + A_2(t), \quad B(t) = A_1(t) + B_1(t) + B_2(t), \\ g(t) &= 1 + \|v(t)\|_\infty^2, \quad f(t) = C\|\nabla_H T(t)\|_2^2, \end{aligned}$$

we have

$$\frac{d}{dt}A(t) + B(t) \leq Cg(t)A(t) + f(t).$$

We are going to show that

$$g(t) \leq C \log(e + B(t)), \quad (4.1)$$

and thus

$$\frac{d}{dt}A(t) + B(t) \leq KA(t) \log(e + B(t)) + f(t),$$

for a positive constant K depending only on h, \mathcal{T} and $\|v_0\|_\infty + \|T_0\|_\infty$. Noticing that $\|f\|_{L^1((0, \mathcal{T}))} \leq C$, for a constant C depending only on h, \mathcal{T} and $\|v_0\|_2 + \|T_0\|_2$, the conclusion follows from the logarithmic type Gronwall inequality, i.e. Lemma 2.5.

We still need to verify (4.1). By Proposition 4.1, Lemma 2.4 and the Sobolev and Poincaré inequalities, we have

$$\begin{aligned} \|v(t)\|_\infty^2 &\leq C \max \left\{ 1, \sup_{q \geq 2} \frac{\|v\|_q^2}{q} \right\} \log(e + \|\nabla_H v\|_6 + \|v\|_6 + \|u\|_2 + \|v\|_2) \\ &\leq C \log(e + \|\nabla_H v\|_{H^1} + \|v\|_{H^1} + \|u\|_2) \\ &\leq C \log(e + \|\nabla_H v\|_2 + \|\nabla \nabla_H v\|_2 + \|u\|_2) \\ &\leq C \log(e + \|\Delta_H v\|_2 + \|\nabla_H u\|_2 + \|u\|_2) \leq C \log(e + B(t)), \end{aligned}$$

verifying (4.1). This completes the proof. \square

We are now ready to prove the global existence part of Theorem 1.1.

Proof of global existence. Let j_ε be the standard modifier, and set $v_0^\varepsilon = v_0 * j_\varepsilon$ and $T_0^\varepsilon = T_0 * j_\varepsilon$. Then we have

$$\begin{aligned} (v_0^\varepsilon, T_0^\varepsilon) &\rightarrow (v_0, T_0), \text{ in } H^1(\Omega), \quad \text{and} \quad \partial_z v_0^\varepsilon \rightarrow \partial_z v_0, \text{ in } L^m(\Omega), \\ \|v_0^\varepsilon\|_{H^1} &\leq \|v_0\|_{H^1}, \quad \|\partial_z v_0^\varepsilon\|_m \leq \|\partial_z v_0\|_m, \quad \text{and} \quad \|v_0^\varepsilon\|_\infty \leq \|v_0\|_\infty, \\ \|T_0^\varepsilon\|_{H^1} &\leq \|T_0\|_{H^1}, \quad \text{and} \quad \|T_0^\varepsilon\|_\infty \leq \|T_0\|_\infty. \end{aligned}$$

Let $(v_\varepsilon, T_\varepsilon)$ be the unique global strong solution to system (2.6)–(2.8), subject to (1.11)–(1.13), with initial data $(v_0^\varepsilon, T_0^\varepsilon)$, as stated in Proposition 2.1.

By Proposition 4.3, for any $\mathcal{T} \in (0, \infty)$, there is a positive constant C depending only on h, \mathcal{T} and $\|v_0\|_{H^1} + \|\partial_z v_0\|_m + \|v_0\|_\infty + \|T_0\|_\infty$, but independent of ε , such that

$$\sup_{0 \leq t \leq \mathcal{T}} (\|\nabla_H v_\varepsilon\|_2 + \|\partial_z v_\varepsilon\|_2^2 + \|\partial_z v_\varepsilon\|_m^m) + \int_0^\mathcal{T} (\|\nabla \nabla_H v_\varepsilon\|_2^2 + \varepsilon \|\partial_z \nabla v_\varepsilon\|_2^2) dt \leq C.$$

Thanks to this estimates, by Propositions 3.1, 3.4 and 3.5, it follows from the Gronwall inequality that

$$\sup_{0 \leq t \leq \mathcal{T}} (\|v_\varepsilon\|_{H^1}^2 + \|T_\varepsilon\|_{H^1}^2) + \int_0^\mathcal{T} (\|\nabla_H v_\varepsilon\|_{H^1}^2 + \|\nabla_H T_\varepsilon\|_{H^1}^2 + \|\partial_t v_\varepsilon\|_2^2 + \|\partial_t T_\varepsilon\|_2^2) \leq C,$$

from which, the same argument as in the proof of local well-posedness at the end of section 3 yields the global existence of strong solutions. This completes the proof. \square

5. APPENDIX: A LOGARITHMIC SOBOLEV INEQUALITY

In this appendix, we prove a logarithmic Sobolev inequality for anisotropic Sobolev functions, that is the following:

Lemma 5.1. *Let $\mathbf{p} = (p_1, p_2, \dots, p_N)$, with $p_i \in (1, \infty)$, and*

$$\sum_{i=1}^N \frac{1}{p_i} < 1.$$

Then we have

$$\begin{aligned} \|F\|_{L^\infty(\mathbb{R}^N)} &\leq C_{N,\mathbf{p},\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|F\|_{L^r(\mathbb{R}^N)}}{r^\lambda} \right\} \\ &\quad \times \log^\lambda \left(\sum_{i=1}^N (\|F\|_{L^{p_i}(\mathbb{R}^N)} + \|\partial_i F\|_{L^{p_i}(\mathbb{R}^N)}) + e \right), \end{aligned}$$

for any $\lambda > 0$.

Proof. We only give the detail proof for the case that the spatial dimension $N \geq 3$, the case that $N = 2$ can be given similarly. Without loss of generality, we can suppose that $|F(0)| = \|F\|_\infty$. Let $\phi \in C_0^\infty(B_1)$, with $\phi \equiv 1$ on $B_{1/2}$, and set $f = F\phi$. Set

$$\alpha_i = \frac{1}{p_i} \in (0, 1), \quad i = 1, 2, \dots, N,$$

and introduce the new variable $y = (y_1, \dots, y_N)$, with

$$y_i = |x_i|^{\alpha_i - 1} x_i, \quad i = 1, 2, \dots, N.$$

Taking μ_i and κ_i as

$$\mu_i = \frac{1 + \sum_{j=1}^N \alpha_j}{1 + \sum_{j=1}^N \alpha_j - 2\alpha_i}, \quad \kappa_i = \frac{p_i \left(1 + \sum_{j=1}^N \alpha_j\right)}{1 - \sum_{j=1}^N \alpha_j},$$

then it is obvious that $\mu_i > 1$ and $\kappa_i > p_i$, and one can easily check that

$$\frac{1}{\mu_i} + \frac{1}{\kappa_i} + \frac{1}{p_i} = 1. \quad (5.1)$$

Setting $\alpha = \sum_{j=1}^N \alpha_j$, then we have

$$\left(\sum_{j=1}^N \alpha_j - \alpha_i\right) \mu_i - \sum_{j=1}^N \alpha_j = (\alpha - \alpha_i) \frac{1 + \alpha}{1 + \alpha - 2\alpha_i} - \alpha = \frac{\alpha_i(\alpha - 1)}{1 + \alpha - 2\alpha_i},$$

from which, noticing that

$$\alpha = \sum_{j=1}^N \alpha_j = \sum_{j=1}^N \frac{1}{p_j} < 1, \quad \text{and} \quad 1 + \alpha - 2\alpha_i > 1 - \alpha_i > 0,$$

we have

$$N + \left(\sum_{j=1}^N \alpha_j - \alpha_i\right) \mu_i - \sum_{j=1}^N \alpha_j < N. \quad (5.2)$$

Set $K_1 = \{(y_1, \dots, y_N) | y_i = |x_i|^{\alpha_i-1} x_i, x \in B_1\}$. Recall that f can be represented in terms of Δf by the Newtonian potential. Recalling (5.1) and (5.2), it follows from integration by parts and the Hölder inequality that, for any $q \geq 3$,

$$\begin{aligned} |f(0)|^q &= C_N \left| \int_{\mathbb{R}^N} \frac{1}{|x|^{N-2}} \Delta (|f|^q(|x|^{\alpha_1-1} x_1, \dots, |x_N|^{\alpha_N-1} x_N)) dx \right| \\ &= C_N \left| \int_{B_1} \frac{1}{|x|^{N-2}} \Delta (|f|^q(|x|^{\alpha_1-1} x_1, \dots, |x_N|^{\alpha_N-1} x_N)) dx \right| \\ &= C_N \left| \int_{B_1} \sum_{i=1}^N \partial_{x_i} \left(\frac{1}{|x|^{N-2}} \right) \partial_{x_i} (|f|^q(y)) dx \right| \\ &= (N-2) C_N q \sum_{i=1}^N \alpha_i \left| \int_{B_1} \frac{|x_i|^{\alpha_i-1} x_i}{|x|^N} |f|^{q-1}(y) |\partial_{y_i} f(y)| dy \right| \\ &= (N-2) C_N \left(\prod_{j=1}^N \alpha_j \right)^{-1} \\ &\quad \times q \sum_{i=1}^N \alpha_i \left| \int_{K_1} \frac{|x_i|^{\alpha_i-1} x_i}{|x|^N} \prod_{j=1}^N |x_j|^{1-\alpha_j} |f|^{q-1}(y) |\partial_{y_i} f(y)| dy \right| \\ &\leq C_{N,p,q} \sum_{i=1}^N \|f\|_{(q-1)\kappa_i}^{q-1} \|\partial_i f\|_{p_i} \left[\int_{K_1} \left(\frac{|x_i|^{\alpha_i}}{|x|^N} \prod_{j=1}^N |x_j|^{1-\alpha_j} \right)^{\mu_i} dy \right]^{\frac{1}{\mu_i}} \end{aligned}$$

$$\begin{aligned}
&\leq C_{N,\mathbf{p}} q \sum_{i=1}^N \|f\|_{(q-1)\kappa_i}^{q-1} \|\partial_i f\|_{p_i} \left[\int_{B_1} \left(\frac{|x_i|^{\alpha_i}}{|x|^N} \right)^{\mu_i} (\prod_{j=1}^N |x_j|^{1-\alpha_j})^{\mu_i-1} dx \right]^{\frac{1}{\mu_i}} \\
&\leq C_{N,\mathbf{p}} q \sum_{i=1}^N \|f\|_{(q-1)\kappa_i}^{q-1} \|\partial_i f\|_{p_i} \left(\int_{B_1} \frac{dx}{|x|^{N+(\sum_{j=1}^N \alpha_j - \alpha_i)\mu_i - \sum_{j=1}^N \alpha_j}} \right)^{\frac{1}{\mu_i}} \\
&\leq C_{N,\mathbf{p}} q \sum_{i=1}^N \|f\|_{(q-1)\kappa_i}^{q-1} \|\partial_i f\|_{p_i}. \tag{5.3}
\end{aligned}$$

With the aid of (5.3), for any $q \geq 3$, noticing that $q^{\frac{1}{q}} \leq C$ and $(q-1)\kappa_i \geq 2$, we deduce

$$\begin{aligned}
|f(0)| &\leq C_{N,\mathbf{p}} \sum_{j=1}^N \|f\|_{(q-1)\kappa_i}^{1-\frac{1}{q}} \|\partial_i f\|_{p_i}^{\frac{1}{q}} \\
&= C_{N,\mathbf{p}} \sum_{i=1}^N \left[\frac{\|f\|_{(q-1)\kappa_i}}{((q-1)\kappa_i)^\lambda} \right]^{1-\frac{1}{q}} [(q-1)\kappa_i]^{\lambda(1-\frac{1}{q})} \|\partial_i f\|_{p_i}^{\frac{1}{q}} \\
&\leq C_{N,\mathbf{p},\lambda} \sum_{i=1}^N \left[\frac{\|f\|_{(q-1)\kappa_i}}{((q-1)\kappa_i)^\lambda} \right]^{1-\frac{1}{q}} q^\lambda \|\partial_i f\|_{p_i}^{\frac{1}{q}} \\
&\leq C_{N,\mathbf{p},\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|f\|_r}{r^\lambda} \right\} q^\lambda \sum_{i=1}^N \|\partial_i f\|_{p_i}^{\frac{1}{q}},
\end{aligned}$$

and thus

$$|f(0)| \leq C_{N,\mathbf{p},\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|f\|_r}{r^\lambda} \right\} \sum_{i=1}^N \inf_{q \geq 3} \left(q^\lambda (\|\partial_i f\|_{p_i} + e^{4\lambda})^{\frac{1}{q}} \right).$$

One can check that

$$\log(\|\partial_i f\|_{p_i} + e^{4\lambda}) \leq \max\{1, 4\lambda\} \log(\|\partial_i f\|_{p_i} + e)$$

and

$$\inf_{q \geq 3} \left(q^\lambda (\|\partial_i f\|_{p_i} + e^{4\lambda})^{\frac{1}{q}} \right) = \left(\frac{e}{\lambda} \right)^\lambda \log^\lambda(\|\partial_i f\|_{p_i} + e^{4\lambda}).$$

Therefore, we have

$$|f(0)| \leq C_{N,\mathbf{p},\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|f\|_r}{r^\lambda} \right\} \sum_{i=1}^N \log^\lambda(\|\partial_i f\|_{p_i} + e).$$

This implies

$$\|F\|_\infty = |f(0)| \leq C_{N,\mathbf{p},\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|f\|_r}{r^\lambda} \right\} \sum_{i=1}^N \log^\lambda(\|\partial_i f\|_{p_i} + e)$$

$$\begin{aligned}
&\leq C_{N,\mathbf{p},\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|F\|_r}{r^\lambda} \right\} \sum_{i=1}^N \log^\lambda(\|\partial_i F\|_{p_i} + \|F\|_{p_i} + e) \\
&\leq C_{N,\mathbf{p},\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|F\|_r}{r^\lambda} \right\} \log^\lambda(\|F\|_{W^{1,\mathbf{p}}} + e),
\end{aligned}$$

proving the conclusion. \square

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REFERENCES

- [1] Azérad, P.; Guillén, F.: *Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics*, SIAM J. Math. Anal., **33** (2001), 847–859.
- [2] Bardos, C.; Lopes Filho, M. C.; Niu, D.; Nussenzveig Lopes, H. J.; Titi, E. S.: *Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking*, SIAM J. Math. Anal., **45** (2013), 1871–1885.
- [3] Bresch, D.; Guillén-González, F.; Masmoudi, N.; Rodríguez-Bellido, M. A.: *On the uniqueness of weak solutions of the two-dimensional primitive equations*, Differential Integral Equations, **16** (2003), 77–94.
- [4] Brézis, H.; Gallouet, T.: *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. **4**(4) (1980), 677–681.
- [5] Brézis, H.; Wainger, S.: *A Note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations **5**(7) (1980), 773–789.
- [6] Cao, C.; Farhat, A.; Titi, E. S. Global well-posedness of an inviscid three-dimensional pseudo-Hasegawa-Mima model. *Comm. Math. Phys.* **319** (2013), 195–229.
- [7] Cao, C.; Ibrahim, S.; Nakanishi, K.; Titi, E. S.: *Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics*, Comm. Math. Physics, **337** (2015), 473–482.
- [8] Cao, C.; Li, J.; Titi, E. S.: *Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity*, Arch. Rational Mech. Anal., **214** (2014), 35–76.
- [9] Cao, C.; Li, J.; Titi, E. S.: *Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity*, J. Differential Equations, **257** (2014), 4108–4132.

- [10] Cao, C.; Li, J.; Titi, E. S.: *Global well-posedness of the 3D primitive equations with only horizontal viscosity and diffusivity*, Comm. Pure Appl. Math., **69** (2016), 1492–1531.
- [11] Cao, C.; Li, J.; Titi, E. S.: *Global well-posedness of the 3D primitive equations with horizontal viscosities and vertical diffusion*, preprint.
- [12] Cao, C.; Titi, E. S.: *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math., **166** (2007), 245–267.
- [13] Cao, C.; Titi, E. S.: *Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion*, Comm. Math. Phys., **310** (2012), 537–568.
- [14] Cao, C.; Wu, Jia. Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation. *Arch. Ration. Mech. Anal.* **208** (2013), no. 3, 985–1004.
- [15] Constantin, P.; Foias, C.: *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.
- [16] Coti Zelati, M.; Huang, A.; Kukavica, I.; Temam, R.; Ziane, M.: *The primitive equations of the atmosphere in presence of vapour saturation*, Nonlinearity, **28** (2015), 625–668.
- [17] Danchin, R.; Paicu, M. Global existence results for the anisotropic Boussinesq system in dimension two. *Math. Models Methods Appl. Sci.* **21** (2011), no. 3, 421–457.
- [18] Guillén-González, F.; Masmoudi, N.; Rodríguez-Bellido, M. A.: *Anisotropic estimates and strong solutions of the primitive equations*, Differ. Integral Equ., **14** (2001), 1381–1408.
- [19] Guo, B; Huang, D.: *Existence of the universal attractor for the 3-D viscous primitive equations of large-scale moist atmosphere*. J. Differential Equations, **251** (2011), 457–491.
- [20] Guo, B.; Huang, D.: *Existence of weak solutions and trajectory attractors for the moist atmospheric equations in geophysics*. J. Math. Phys., **47** (2006), 23pp.
- [21] Haltiner, G.; Williams, R.: *Numerical Weather Prediction and Dynamic Meteorology*, second ed., Wiley, New York, 1984.
- [22] Hieber, M.; Kashiwabara, T.: *Global well-posedness of the three-dimensional primitive equations in L^p -space*, Arch. Rational Mech. Anal., **221** (2016), 1077–1115.
- [23] Hieber, M.; Hussein, A.; Kashiwabara, T.: *Global strong L^p well-Posedness of the 3D primitive equations with heat and salinity diffusion*, arXiv:1605.02614.
- [24] Kobelkov, G. M.: *Existence of a solution in the large for the 3D large-scale ocean dynamics equations*, C. R. Math. Acad. Sci. Paris, **343** (2006), 283–286.
- [25] Kukavica, I.; Pei, Y.; Rusin, W.; Ziane, M.: *Primitive equations with continuous initial data*, Nonlinearity, **27** (2014), 1135–1155.

- [26] Kukavica, I.; Ziane, M.: *The regularity of solutions of the primitive equations of the ocean in space dimension three*, C. R. Math. Acad. Sci. Paris, **345** (2007), 257–260.
- [27] Kukavica, I.; Ziane, M.: *On the regularity of the primitive equations of the ocean*, Nonlinearity, **20** (2007), 2739–2753.
- [28] Ladyzhenskaya, O. A.: *The mathematical theory of viscous incompressible flow*, Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2 Gordon and Breach, Science Publishers, New York-London-Paris 1969.
- [29] Lewandowski, R.: *Analyse Mathématique et Océanographie*, Masson, Paris, 1997.
- [30] Li, J.; Titi, E. S.: *Global well-posedness of the 2D Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal., **220** (2016), 983–1001.
- [31] Li, J.; Titi, E. S.: *Global well-posedness of strong solutions to a tropical climate model*, Discrete Contin. Dyn. Syst., **36** (2016), 4495–4516.
- [32] Li, J.; Titi, E. S.: *A tropical atmosphere model with moisture: global well-posedness and relaxation limit*, arXiv:1507.05231 [math.AP].
- [33] Li, J.; Titi, E. S.: *Recent advances concerning certain class of geophysical flows*, arXiv:1604.01695 [math.AP].
- [34] Li, J.; Titi, E. S.: *Existence and uniqueness of weak solutions to viscous primitive equations for certain class of discontinuous initial data*, arXiv:1512.00700 [math.AP].
- [35] Li, J.; Titi, E. S.: *Small aspect ratio limit from Navier-Stokes equations to primitive equations: mathematical justification of hydrostatic approximation*, preprint.
- [36] Li, J.; Xin, Z.: *Local well-posedness and blow-up criteria of strong solutions to the Ericksen-Leslie system in bounded domains of \mathbb{R}^3* , preprint.
- [37] Lions, J. L.; Temam, R.; Wang, S.: *New formulations of the primitive equations of the atmosphere and applications*, Nonlinearity, **5** (1992), 237–288.
- [38] Lions, J. L.; Temam, R.; Wang, S.: *On the equations of the large-scale ocean*, Nonlinearity, **5** (1992), 1007–1053.
- [39] Lions, J. L.; Temam, R.; Wang, S.: *Mathematical study of the coupled models of atmosphere and ocean (CAO III)*, J. Math. Pures Appl., **74** (1995), 105–163.
- [40] Majda, A.: *Introduction to PDEs and Waves for the Atmosphere and Ocean*, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
- [41] Pedlosky, J.: *Geophysical Fluid Dynamics, 2nd edition*, Springer, New York, 1987.
- [42] Petcu, M.; Temam, R.; Ziane, M.: *Some mathematical problems in geophysical fluid dynamics*, Elsevier: Handbook of Numerical Analysis, **14** (2009), 577–750.
- [43] Simon, J.: *Compact sets in the space $L^p(0, T; B)$* , Ann. Mat. Pure Appl., **146** (1987), 65–96.
- [44] Tachim-Medjo, T.: *On the uniqueness of z -weak solutions of the three-dimensional primitive equations of the ocean*, Nonlinear Anal. Real World Appl.,

- 11** (2010), 1413–1421.
- [45] Temam, R.: *Navier-Stokes Equations. Theory and Numerical Analysis*, Revised edition, Studies in Mathematics and its Applications, 2., North-Holland Publishing Co., Amsterdam-New York, 1979.
 - [46] Vallis, G. K.: *Atmospheric and Oceanic Fluid Dynamics*, Cambridge Univ. Press, 2006.
 - [47] Washington, W. M.; Parkinson, C. L.: *An Introduction to Three Dimensional Climate Modeling*, Oxford University Press, Oxford, 1986.
 - [48] Wong, T. K.: *Blowup of solutions of the hydrostatic Euler equations*, Proc. Amer. Math. Soc., **143** (2015), 1119–1125.
 - [49] Zeng, Q. C.: *Mathematical and Physical Foundations of Numerical Weather Prediction*, Science Press, Beijing, 1979.

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